UNIVERSITY
OF MICHIGAN
JUL 27 1959
MATHEMATICS
LIBRARY

Mathematical Reviews

Edited by

R. P. Boas, Jr. J. L. Doob W. S. Massey

S. H. Gould, *Executive Editor*

Chandler Davis, W. Freiberger *and* J. A. Zilber, *Associate Editors*

H. A. Pogorzelski, *Copy Editor*

Vol. 20, No. 2 February, 1959 Reviews 814–1620

TABLE OF CONTENTS

Logic and Foundations 131
Set Theory 132
Combinatorial Analysis 132
Order 132
General Algebraic Systems 133

Theory of Numbers 133
Commutative Rings and Algebras 136
Algebraic Geometry 138
Linear Algebra 142
Associative Rings and Algebras 142
Non-Associative Rings and Algebras 143
Homological Algebra 143

Groups and Generalizations 144
Topological Groups and Lie Theory 149
Topological Algebra 153

Functions of Real Variables 153
Measure and Integration 154
Functions of a Complex Variable 156
Functions of Several Complex Variables, Complex Manifolds 163
Special Functions 165
Ordinary Differential Equations 166
Partial Differential Equations 173
Differential Algebra 184
Potential Theory 184
Finite Differences and Functional Equations . . . 186

Sequences, Series, Summability 188
Approximations and Expansions 189
Fourier Analysis 191
Integral Transforms 194
Integral and Integrodifferential Equations 195

Functional Analysis 196
Calculus of Variations 208

Geometries, Euclidean and Other 209
Convex Sets and Distance Geometries 212
General Topology, Point Set Theory 214
Algebraic Topology 216
Differential Geometry, Manifolds 220

Probability 223
Statistics 228

Numerical Methods 231
Computing Machines 236

Mechanics of Particles and Systems 238
Statistical Thermodynamics and Mechanics . . . 239
Elasticity, Plasticity 240
Structure of Matter 243
Fluid Mechanics, Acoustics 245
Optics, Electromagnetic Theory, Circuits 250
Classical Thermodynamics, Heat Transfer 252
Quantum Mechanics 252
Relativity 256
Astronomy 258
Geophysics 261

Operations Research and Econometrics 261
Biology and Sociology 261
Information and Communication Theory 262
Control Systems 263

History and Biography 264
General 265

AUTHOR INDEX

Abhyankar, Shreeram 872
Abrikosov, A. A. 1546
Adachi, M. 1317
Adel'son-Vel'skiĭ, G. M. 1238
Adler, C. F. 1249
Afriat, S. N. 1389
Agostinelli, C. 1514
Ahiezer, N. I. 971
Ahmedov, K. T. 1241
Aitchison, J. 1382
Aiserman, M. A. 1037
Akim, E. L. 998
Aksel'rad, Ė. L. 1484
Alaci, V. 1100
Albrecht, R. 1136
Aleksandrov, A. D. 1105, 1301
Aleksandrov, I. A. 978
Aleksandrov, P. S. 1613
Allahverdiev, D. Ė. 1227
Allen, D. N. de G. 1460
Altmann, S. L. 1550, 1551
Ambarcumyan, S. A. 1470
Andreev, A. F. 1015
Angelescu, T. 1332
Angelitch, T. P. 1461
Anonymous
Obituary: Vasiliĭ Vasil'evič Popov. 1611
Antoniu, M. 1442
Apostol, T. M. 838
Aragnol, A. 1331a-d
Arens, R. 1169
Arima, S. 846
Arscott, F. M. 1001
Artmann, K. 1553
Atsuji, M. 1290
Azrilyant, P. A. 1528
Babiĭ, V. P. 1258
Babuška, I. 1458
Bachelis, R. D. 1063
Backus, G. 1512
Bailey, N. T. J. 1592
Bakalyaev, O. S. 1255
Baker, G. A., Jr. 1533
Balcerzyk, S. 935
Bambah, R. P. 1284
Bandić, I. 1010, 1456
Banditch, I. = Bandić, I.
Baskakov, V. A. 1153
Bass, H. 827
Bass, J. 1500
Bauer, F. L. 1409
Bedel'baev, A. K. 1601
Belkina, M. G. 1528
Bellman, R. 1073
Beneš, M. 1393, 1394
Benthem, C. W. 1181
Benz, W. 1256, 1257
Berge, C. 1323
Berkes, J. 1254
Berlinkov, M. L. 826, 917
Bernstein, R. I. 1371
Bespamyatnih, N. D. 1609
Bhatnagar, P. L. 1493, 1516
Bicadze, A. V. 962, 1079, 1087
Biedenharn, L. C. 1543
Biermann, K. R. 1608
Bihari, I. 1031
Bilimović, A. 1250
Bitsadze, A. V. = Bicadze, A. V.
Bizadze, W. A. = Bicadze, A. V.
Blumenson, L. E. 1371
Blyumenfel'd, V. N. 1085
Boas, R. P., Jr. 972, 984
Bocchieri, P. 1448
Bogoliubow, N. N. = Bogolyubov, N. N.
Bogolyubov, N. N. 1535, 1567
Bojanić, R. 1064
Bolt, B. A. 1443
Boltyanskiĭ, V. G. 1312
Bonsall, F. F. 1194
Boole, G. 1123
Borok, V. M. 1104
Bouligand, G. 814
Bourgin, D. G. 1297
Box, G. E. P. 1415
Braakman, T. C. 1181
Bradeanu, P. 1441
Braier, A. 1442
Bram, J. 1152
Brearley, M. N. 1444
Brehovskih, L. M. 1476
Breny, H. 1360
Broadwell, J. E. 1486
Brodersen, S. 1552
Brodskiĭ, M. S. 1229
Browder, F. E. 1064
Bruschi, M. 1218
Buchdahl, H. A. 1570
Buchholz, H. 1002
Buck, R. C. 984, 1278
Burgoyne, N. 1538b
Burkill, H. 1167
Burkill, J. C. 1215
Butzer, P. L. 1232
Bychawski, Z. 1468
Campanato, S. 1093
Capriz, G. 1474
Carini, G. 1515
Carr, J. W., III. 1417
Cartan, H. 1339
Cartier, P. 924
Cartwright, D. E. 1396
Cassina, U. 1605
Čebotarev, A. S. 1342
Čencov, N. N. 1366
Černikov, S. N. 920
Černý, V. 1430
Čerskiĭ, Yu. I. 1182
Červená, A. 1274
Česari, L. 954
Četaev, N. G. 1416
Chang, Bomshik 906
Chang, C. C. 820, 821
Châtelet, A. 856
Chen, Kien-Kwong 1141
Chisini, O. 864
Choquet, G. 1119
Chow, Wei-Liang 871
Ciobanu, G. 1440
Ciorănescu, N. 997
Clark, C. E. 1390
Clayden, W. A. 1491
Cohan, N. V. 999
Coles, W. J. 848
Conrad, P. 887
Conte, L. 1606
Corduneanu, C. 1032, 1042
Corson, H. H. 1292
Cotlar, M. 1218
Coutie, G. A. 1415
Cowling, T. G. 1511
Cox, A. D. 1491
Cox, D. R. 1399
Cramariuc, R. 1525
Császár, Á. 1289
Csonka, P. 1466
Culick, F. E. C. 1502
Curry, H. B. 817
Curtis, C. W. 933

(Contined on cover 3)

Journal references in Mathematical Reviews are now given in the following form: J. Brobdingnag. Acad. Sci. (7) 4(82) (1952/53), no. 3, 17–42 (1954), where after the abbreviated title one has: (series number) volume number (volume number in first series if given) (nominal date), issue number if necessary, first page–last page (imprint date). In case only one date is given, this will usually be interpreted as the nominal date and printed immediately after the volume number (this is a change from past practice in Mathematical Reviews where a single date has been interpreted as the imprint date). If no volume number is given, the year will be used in its place.

Reviews reprinted from Applied Mechanics Reviews, Referativnyĭ Žurnal, or Zentralblatt für Mathematik are identified in parentheses following the reviewer's name by AMR, RŽMat (or RŽMeh, RŽAstr. Geod.), Zbl, respectively.

MATHEMATICAL REVIEWS

Published monthly, except August, by
THE AMERICAN MATHEMATICAL SOCIETY, 190 Hope St., Providence 6, R.I.

Sponsored by

THE AMERICAN MATHEMATICAL SOCIETY
THE MATHEMATICAL ASSOCIATION OF AMERICA
THE INSTITUTE OF MATHEMATICAL STATISTICS
THE EDINBURGH MATHEMATICAL SOCIETY
SOCIÉTÉ MATHÉMATIQUE DE FRANCE
DANSK MATEMATISK FORENING
HET WISKUNDIG GENOOTSCHAP TE AMSTERDAM
THE LONDON MATHEMATICAL SOCIETY
POLSKIE TOWARZYSTWO MATEMATYCZNE
UNIÓN MATEMÁTICA ARGENTINA
INDIAN MATHEMATICAL SOCIETY
UNIONE MATEMATICA ITALIANA
THE SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS

Editorial Office
MATHEMATICAL REVIEWS, 190 Hope St., Providence 6, R.I.

Subscription: Price $50 per year ($25 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions should be addressed to the American Mathematical Society, 190 Hope St., Providence 6, R.I.

The preparation of the reviews appearing in this publication is made possible by support provided by a grant from the National Science Foundation. The publication was initiated with funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers or proprietors of the publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Published for the American Mathematical Society by N.V. Erven P. Noordhoff, Uitgeversmaah, Groningen, the Netherlands.

Mathematical Reviews

Vol. 20, No. 2 **FEBRUARY, 1959** **Reviews 814-1620**

LOGIC AND FOUNDATIONS

See also 1294, 1399, 1602, 1603.

814:

***Bouligand, Georges; et Desgranges Jean. Le déclin des absolus mathématico-logiques.** Collection "Esprit et Méthode". Société d'Edition d'Enseignement Supérieur, Paris, 1949. 270 pp. 600 francs.

This is a book, on an advanced level, about the philosophical treatment of mathematical problems and results. The book is permeated by a realization of the fragility of our most firmly based certainties in mathematics. Its aim is to show the non-existence of any absolute knowledge about the diverse parts of mathematics at any level of considering them. So it is emphasized that ordinary algebra (of the complex numbers), which seemed at the beginning to represent the only algebra, has lost this absolute place and has become only one of the algebras (of hypercomplex numbers). A second instance is given by analysis, which progresses by continually comprehending in itself new infinite processes and always remains what Bouligand calls an "open system", i.e., a system not allowing itself to be described by norms fixed in advance. Logic also is no longer an absolute basis for mathematics, as, e.g., Russell believed.

The book consists of four parts, well integrated with one another, although they are written by two authors: An epistemological glance, by Bouligand; A historical-philosophical inquiry, by Desgranges; Some didactical views, by Bouligand; and Index, with comments, of some terms of the actual philosophical-mathematical language, by Desgranges.

In Bouligand's parts of the book a principal tool of the whole presentation is the system: problems and global synthesis. The main task of the mathematician is to solve problems, i.e., given certain sets, to extract elements fulfilling certain conditions. But the exact formulation of a problem requires a precise form, which can be reached only in the frame of global synthesis, where the different categories, e.g., points, straight lines, circles, etc., are fixed. There is a mutual interdependence between the problems and the global synthesis. The wisest attitude in mathematics is therefore to reserve to oneself a certain independence in judging different situations, a spirit which F. Gonseth has called "idoneism".

Desgranges gives an historical review of the development of mathematical philosophy. The two main concepts with which he works are the predicative system and the dialectical conception of mathematics. The first is the view that mathematics, as a science, is founded on an assured and immovable basis. The second is the view that mathematics undergoes a steady development, changing its appearance totally from time to time. Desgranges passes in review the predicative philosophers: Plato, Descartes, Leibniz, Kant, Liard, Hermite. He reckons among them also Russell and Hilbert. Axiomatics is, according to him, a predicative theory. In the second chapter he passes to the philosophers who have taken the view that mathematics is also an experimental science. They form the transition to the dialectical philosophers. The "experimental" philosophers are F. Bacon, Hume, d'Alembert, Cournot, A. Comte, E. Borel. In the third chapter the author presents the views of M. Winter, L. Brunschvicg, P. Boutroux and of the most modern philosophers F. Gonseth, A. Lautmann, G. Casanova, M. Fréchet and G. Bouligand. Some of these modern thinkers have supplied a truly general conception of mathematical science. For example, Gonseth says that mathematics can be only seemingly detached from its intuitive foundations; its method manifests itself in the dialectical organization of cognition. Bouligand, too, approves without reserve the tendency of "asking nature, with readiness to receive its teachings and not to dictate to it its answers." The word "dialectics" means the tendency to refuse all the mathematicological absolutes dear to the predicative doctrines. According to Desgranges, the mathematical philosophy of Bouligand takes up the same position as Gonseth's, but avoids its defects.

B. Germansky (Berlin)

815:

***Jevons, W. Stanley. The principles of science: a treatise on logic and scientific method. With a new introduction by Ernest Nagel.** Dover Publications, Inc., New York, 1958. liii+786 pp. $2.98.

An unchanged reproduction of the second edition, of 1877 (the first edition is of 1874), with an introduction by Nagel written in 1957.

816:

***Saminsky, Lazare. Physics and metaphysics of music and essays on the philosophy of mathematics.** Martinus Nijhoff, The Hague, 1957. vii+151 pp. 10.45 guilders.

The book consists of four essays: Physics and metaphysics of music, The roots of arithmetic, Critique of new geometrical abstractions, The philosophical value of science.

817:

***Curry, Haskell B.; and Feys, Robert. Combinatory logic. Vol. I. With two sections by W. Craig.** Studies in logic and the foundations of mathematics. North-Holland Publishing Co., Amsterdam, 1958. xvi+417 pp. 42.00 guilders.

This book is the first volume of a projected two-volume treatise on combinatory logic. Combinatory logic is concerned with certain basic notions of the foundations of mathematics which are usually used in an intuitive and unanalysed way. Such notions include substitution, usually introduced by the use of variables, and classification of the entities of a system into types, which is usually provided for by rules which are auxiliary to, but not part of, the system. The part of combinatory logic which is concerned with questions of a fundamental nature which, like substitution, involve variables, is

called the theory of combinators. Descriptions of parts of this theory are available in a number of papers by Curry and Feys, Kleene, Rosser, Rosenbloom, the reviewer, and others. The material in this book recapitulates and organizes the central results, and furnishes some new ones. This part of combinatory logic is closely allied to some of the theories of lambda conversion due to Church.

The second part of combinatory logic, called illative combinatory logic, is concerned with problems of classifying entities into categories. Illative combinatory logic is closely related to logics of propositions, relations, and sets, and relatively few results within it are available in the literature. The material on illative combinatory logic in the present book represents the results of both reported and heretofore unreported research, together with new formulations of illative theories and theorems which concern them. Many of these concern questions of consistency and studies of the character of deductions in these systems.

The material is divided into ten chapters, each of which is provided with historical comments which trace the origins and development of the ideas presented. The first five chapters are informal, largely, and descriptive. Chapter 1 is devoted to formal systems in general, and Chapter 2 to epitheory. Chapter 3 describes that part of the theory of lambda conversion germane here, and the Church-Rosser theorem is developed in Chapter 4. In Chapter 5, the theory of combinators is described in an intuitive way and motivations are provided for its study. The synthetic theory of combinators is presented initially in Chapter 6, and developed further in Chapter 7, which is devoted to the logistic foundations. The last three chapters are devoted to that part of illative combinatory logic which concerns the theories of functionality. Additional investigation of illative combinatory logic, and of combinatory arithmetic, have been planned for Volume II.

The reader is provided with a list of basic constants in Appendix A, and with a list of some properties of relations which are used throughout the work in Appendix B. A comprehensive bibliography and an index are furnished. A list of errata is provided with the book by the publisher, and a supplementary list has been prepared by the authors. In addition, on page 365, condition B4, the last occurrence of T' should be $\mathfrak{T}'$.

E. J. Cogan (Bronxville, N.Y.)

818:

Rose, Alan; and Rosser, J. Barkley. Fragments of many-valued statement calculi. Trans. Amer. Math. Soc. **87** (1958), 1–53.

The set $\mathscr{T}$ of truth values of a many-valued statement calculus is assumed to contain numbers in the interval $[0, 1]$ and to be closed under the operations $c(x, y) = \min(1, 1-x+y)$ and $n(x) = 1-x$. In addition, a number S in the interval is chosen: truth values $x > S$ are called designated, and truth-values $x < S$ are called undesignated. The value 0 is always undesignated, 1 is always designated, and S, if it is a truth-value, may be either designated or not. The problem to which the authors address themselves is, once $\mathscr{T}$, S, and a set of undefined truth functions are chosen, to provide sets of axioms so that a formula is provable by means of a rule like modus ponens (or modus ponens itself) if and only if the only values the formula takes on are designated. Cases in which $\mathscr{T}$ is finite or infinite are distinguished, as well as cases in which $S = 1$ and $S < 1$. Distinctions are also made as to which truth functions are taken as undefined. Section 11 is devoted to some results from the theory of linear polynomials provided by Theodor Motzkin. A mapping between linear polynomials and formulas of a many-valued statement calculus is used to get results for systems in which $\mathscr{T}$ is infinite.

E. J. Cogan (Bronxville, N.Y.)

819:

Meredith, C. A. The dependence of an axiom of Łukasiewicz. Trans. Amer. Math. Soc. **87** (1958), 54.

This is a proof of A5 of the Rose and Rosser paper reviewed above from A1–A3 and (3.51) of that paper.

E. J. Cogan (Bronxville, N.Y.)

820:

Chang, C. C. Proof of an axiom of Łukasiewicz. Trans. Amer. Math. Soc. **87** (1958), 55–56.

This is a proof of A5 of the Rose and Rosser paper reviewed above from A1–A4 of that paper.

E. J. Cogan (Bronxville, N.Y.)

821:

Chang, C. C. Algebraic analysis of many valued logics. Trans. Amer. Math. Soc. **88** (1958), 467–490.

This is an algebraization of a logical theory, the many-valued (MV) propositional calculus. The central theme of the paper is an attempt to find a proof of the completeness of the $\aleph_0$-valued logic (developed by Łukasiewicz and Tarski [C. R. Soc. Sci. Varsovie **23** (1930), 30–50] and by Rose and Rosser [≠ 818 above]) by using algebraic results concerning MV-algebras, which are defined and studied here. The completeness theorem is not proved, but the completeness of the $\aleph_0$-valued logic is shown to be equivalent to semi-simplicity (i.e., intersection of maximal ideals is $\{0\}$) of an algebra L which is defined in a natural way from the axioms of Łukasiewicz.

B. A. Galler (Ann Arbor, Mich.)

SET THEORY

822:

Stupina, I. D. On some properties of Γ-operations. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 329–348. (Russian)

A general theorem of Kozlova [same Izv. **21** (1957), 349–370; MR **19**, 829] on imbedding of sets is shown to be valid [Σ-consistent] [cf. Novikov, Trudy Mat. Inst. Steklov **38** (1951), 279–316; MR **14**, 234] if S is the system of CA_2-sets [CA_n-sets], N is a rigid basis of a Γ-operation [Alexandroff, Fund. Math. **5** (1924), 160–165], and U is the property of being, for example, an N-enumerable-fold point.

F. Bagemihl (Notre Dame, Ind.)

COMBINATORIAL ANALYSIS

See 1596.

ORDER

See also 821, 915.

823:

Wallace, Alexander Doniphan. The center of a compact lattice is totally disconnected. Pacific J. Math. **7** (1957), 1237–1238.

The author proves that the center of any topological

lattice, with compact Hausdorff topology, is totally disconnected. *G. Birkhoff* (Cambridge, Mass.)

824:

Wallace, Alexander Doniphan. Two theorems on topological lattices. Pacific J. Math. **7** (1957), 1239–1241.

If a convex subset A of a connected topological lattice L disconnects L, then the complement of A consists of two connected components: the elements above A and those below A. If L is also compact, and A is open or closed, then A is connected. If, further, L is metrizable, then Whyburn's theory of cyclic elements is applicable.

G. Birkhoff (Cambridge, Mass.)

825:

Wallace, Alexander Doniphan. Factoring a lattice. Proc. Amer. Math. Soc. **9** (1958), 250–252.

Let a topological lattice L be homeomorphic with a closed square. It is shown that, if no maximal chain cuts L into more than two components, then L is isomorphic with $I \times I$. The first assumption could be replaced by the hypotheses that L is compact, connected, imbeddable in the plane, and without cutpoints.

G. Birkhoff (Cambridge, Mass.)

826:

Berlinkov, M. L. On σ-lattices which are compact with respect to (o)-convergence. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 2(74), 123–127. (Russian)

A set S is called a σ-lattice if every sequence $\{x_n\}$ of its elements defines an intersection $\bigcap x_n$ and a union $\bigcup x_n$. A sequence of elements of a σ-lattice is called (o)-convergent (briefly, convergent) if the limits

$$\liminf x_n = \bigcup_{n=1}^{\infty} \{ \bigcap_{k=n}^{\infty} x_k \}, \quad \limsup x_n = \bigcap_{n=1}^{\infty} \{ \bigcup_{k=n}^{\infty} x_k \}$$

coincide. A σ-lattice is called compact with respect to (o)-convergence (briefly, compact) if every sequence of elements contains a convergent subsequence. Theorem 1: A compact σ-lattice S cannot have a sublattice which is isomorphic to N (the non-negative integers with $x \cap y = \gcd(x,y)$, $x \cup y = \mathrm{lcm}(x,y)$).

An element x of a σ-lattice S is σ-finite, if for every sequence x_n of S such that $x \leqq \bigcup x_n$, the same relation also holds with $\bigcup$ running over some finite subsequence. A set M of elements of the lattice S is called continuous if every subset of M has a union and if the relation $a \leqq \bigcup_{x \in R} x$ is not satisfied for any element a or subset R of M, unless R contains a. Theorem 2: If M is a continuous subset of a compact lattice S, and M consists of σ-finite elements, then it has power less than the power of the continuum.

The author defines the direct product of a set of σ-lattices and proves that the direct product of a finite or countable set of compact σ-lattices is a compact σ-lattice. However, if the set has the power of the continuum (or higher), and each lattice has at least two elements, the direct product P of all these lattices is not compact.

Theorem 5: Let S be a σ-lattice; A, a subset such that every element of S is a union of elements of A. Let all elements of A be σ-finite; let A contain a subsequence a_m of elements which are distributive (a_m, y, z satisfy the distributive law for every y, z in S). Let all the elements below a_m form a compact lattice for every m; let A be contained in the union of these compact lattices. Then S is a compact lattice. *J. L. Brenner* (Palo Alto, Calif.)

827:

Bass, Hyman. Finite monadic algebras. Proc. Amer. Math. Soc. **9** (1958), 258–268.

Following Halmos [Compositio Math. **12** (1956), 217–249; MR **17**, 1172], a "monadic algebra" is a Boolean algebra with distributive closure operation, in which the complement of any closed element is closed. The finite monadic algebras are described explicitly, and shown to correspond one-one to finite sets of positive integers. The free monadic algebra with r generators is also determined explicitly; it has 2^{r+2^r-1} atoms.

G. Birkhoff (Cambridge, Mass.)

GENERAL ALGEBRAIC SYSTEMS

See 897, 923.

THEORY OF NUMBERS

See also 867, 879, 880, 991a-b, 1284, 1429.

828:

*** Hardy, G. H.; und Wright, E. M. Einführung in die Zahlentheorie. Übersetzt von H. Ruoff.** R. Oldenbourg, Münich, 1958. xvi+480 pp. DM 74.00.

A translation of the third edition [An introduction to the theory of numbers, Oxford, 1954; MR **16**, 673].

829:

*** Hua, L. K. Su lung tao yeng. [Introduction to number theory.]** Science Publishing Co., Peking, 1957. xvi+652 pp. 9.00 yuan; paperbound, 6.30 yuan.

This is a valuable and important textbook on number theory, somewhat on the lines of Hardy and Wright [An introduction to the theory of numbers, Oxford, 1954; MR **16**, 673], but going far beyond it in scope. The text is very clear, in an easy form of literary Chinese, and might serve as an excellent introduction to mathematical Chinese. A Chinese-English-Russian vocabulary of some 12 pages at the end of the book will be of particular help to the beginner of such studies. The book contains many recent results, useful tables, e.g., of quadratic fields, and problems for the student. It should lead to much work in number theory among Chinese students.

Contents: 1. Prime factorisation. 2. Congruences. 3. Quadratic residues. 4. Properties of polynomials. 5. Distribution of primes. 6. Arithmetic functions. 7. Trigonometrical sums and character sums. 8. Arithmetical questions related to elliptic modular functions. 9. The prime number theorem (proofs by Ikehara and Selberg; theorem of Dirichlet). 10. The approximation of real numbers by means of continued fractions. Uniform distribution mod 1. 11. Indeterminate equations. 12. Binary quadratic forms (Siegel's theorem on the class number). 13. Modular transformations (geometrical theory of binary quadratic forms). 14. Integral matrices and their applications. 15. p-adic numbers (valuation theory). 16. Algebraic number fields. 17. Algebraic and transcendental numbers (theorems of Roth, Hermite, Lindemann, Gel'fond-Schneider). 18. Problems of Waring and Prouhet-Tarry. 19. Schnirelman densities. 20. Geometry of numbers.

This short list does scarcely justice to the richness of material dealt with.

K. Mahler (Manchester)

830:

***Michel, P.-H. Les nombres figurés dans l'arithmétique Pythagoricienne.** Les conférences du Palais de la Découverte. Histoire des Sciences (Série D). Université de Paris, Paris, 1958. 23 pp.

In a survey of Pythagorean number theory the author comes to the conclusion that Nichomachus' theorem on heteromecic (oblong) numbers of the form $n(n+1)$ is capable of a generalization for "solid" heteromecic numbers of the form $n(n+1)(n+2)$, representing them too as sums of consecutive even numbers. Here he quotes the English edition of Nicomachus' "Introduction to arithmetic" [Macmillan, New York, 1926], where the generalization of the theorem on square and cubic numbers by L. C. Karpinski and N. Anning is given [pp. 289–290]. An example of the decomposition of such a "solid" heteromecic number is $12+14+16+18=60=3\cdot4\cdot5$. The author concludes that the Pythagorean school was not quite as esoteric as it sometimes is thought to be, since it can yield quite positive results.

S. R. Struik (Cambridge, Mass.)

831:

Vinogradov, A. I.; and Linnik, Yu. V. Estimate of the sum of the number of divisors in a short segment of an arithmetic progression. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 277–280. (Russian)

Suppose that $(l, D)=1$ and $0<l<D$. Let $S=\sum_{nD+l\leq x}\tau(nD+l)$, where $\tau(m)$ denotes the number of divisors of m. An upper bound for S is obtained when $D\leqq x^{1-\alpha}$, where α is fixed and $0<\alpha<\frac{1}{2}$, namely

$$(*)\qquad S<C(\alpha)\frac{x}{D}\log\frac{x}{D}\prod_{p\mid D}(1-p^{-1}),$$

where $C(\alpha)$ depends on α. The same expression, with a different $C(\alpha)$, also gives a lower bound for S, as is easily seen. The proof of (*) is technically elementary, using division into several cases. The question is raised: how should $C(\alpha)$ behave as $\alpha\to0$?

H. Davenport (London)

832:

Robinson, Raphael M. Some factorizations of numbers of the form $2^n\pm1$. Math. Tables Aids Comput. **11** (1957), 265–268.

The author has prepared a factorization routine for searching for possible factors of N which lie below 2^{35} and belong to a given arithmetic progression. It is coded for an IBM 701. This he has applied to more than 30 previously unfactored numbers dividing $2^n\pm1$.

No Fermat number has a factor $<2^{32}$ other than those previously known. New factors were discovered for $2^{95}-1$, $2^{109}-1$, $2^{157}-1$ and for 2^n+1 for $n=71$, 109, 112, 113 and 134 as a result of a search for factors $<2^{30}$. The numbers $2^{71}+1$, $2^{112}+1$, $2^{113}+1$, $2^{134}+1$ are now completely factorized together with $2^{95}-1$ whose incomplete factorization was claimed to be complete by Kraitchik [Scripta Math. **18** (1952), 39–52; MR **14**, 121]. There is given a list of values of n for which the complete factorization of 2^n-1 or 2^n+1 has been claimed. Some are marked as doubtful. The list includes $2^{97}-1$, $2^{101}+1$ and $2^{129}-1$, whose factors are unknown to the reviewer.

According to a conjecture, 2^n-1 is a prime if $n=2^m-1$ is a prime. The falsity of this for $m=13$ was established by D. J. Wheeler by a very long machine calculation. The author offers two easily verifiable new counter-examples, namely $m=17$ and 19. The corresponding values of 2^n-1 are divisible by $1768(2^{17}-1)+1$ and $120(2^{19}-1)+1$, respectively.

D. H. Lehmer (Berkeley, Calif.)

833:

Smith, Herschel F. On a generalization of the prime pair problem. Math. Tables Aids Comput. **11** (1957), 249–254.

The author asks what is the smallest range of natural numbers that can contain n primes. He provides tables in partial answer that give sets of arithmetical progressions in which these n primes must lie. For $n=2(1)26$ is given, in effect, one or more sets of n arithmetic progressions $mx+b_i$ $(i=1(1)n)$, for which b_n-b_1 is as small as possible and for which an x might exist making all n terms prime. The existence of such an x is left uncertain for $n>10$. Thus for $n=19$ there may well be a set of 76 consecutive integers containing 19 primes. A shorter range will certainly not suffice. These 19 primes may belong to one of only two sets of arithmetic progressions that have differences 510510. A search among the first $2\cdot10^9$ integers fails to reveal any such cluster of 19 primes.

Four primes can be compressed into a range of length 8 and this happens 549 times below 5073379. Similar surveys for $n=7$, 8, 9, 10 were made as far as 157131419. No second instance to match the famous cluster of 10 primes beginning with 11 was discovered. However a half-dozen clusters of 9 primes were found, one as far away as 74266249.

D. H. Lehmer (Berkeley, Calif.)

834:

Egerváry, E. Auflösung eines homogenen linearen diophantischen Gleichungssystems mit Hilfe von Projektormatrizen. Publ. Math. Debrecen **4** (1956), 481–483.

Let small Roman letters represent n-partite vectors whose components are rational integers, and let, further, $a\cdot x$ stand for the inner product $a_1x_1+a_2x_2+\cdots+a_nx_n$. The greatest common divisor of the n integers $a_1, a_2, \cdots, a_n$ is denoted by $\Delta(a)$. It was shown by Giundice [Giorn. Mat. Battaglini (2) **5(36)** (1898), 225–232] that the general solution x of the equation $a\cdot x=0$ can be expressed in terms of a and $n-1$ integral parameters. A simpler expression for x was given by Barnett and Mendel [Amer. Math. Monthly **49** (1942), 157–170; MR **3**, 268]. A system of m linear homogeneous equations $a^{(i)}\cdot x=0$, $i=1, 2, 3, \cdots, m$, $m<n$, $\Delta(a^{(i)})=1$, was completely solved by Heger [Denkschr. Akad. Wiss. Wien. Math.-Nat. Cl. **14** (1858), Abt. 2, 1–122], who proved that the general solution x can be expressed in terms of $a^{(i)}$ and $n-m$ integral parameters, if the equations are linearly independent. The author gives a short proof of this theorem and a simple expression for the general solution. When $m=1$, this solution is the same as that of Barnett and Mendel. The author makes use of a method more thoroughly dealt with in his paper: "Rank-diminishing operations and the solution of linear equations by finite iteration" (to be published in Acta Sci. Math. Szeged).

W. Ljunggren (Bergen)

835:

Gloden, A. Note d'analyse Diophantienne sur l'équation biquadratique $x_1^4+x_2^4+x_3^4=y_1^4+y_2^4+y_3^4$. Bul. Inst. Politech. Iaşi **4** (1949), 54–57.

The author discusses an elementary method for finding solutions of the equation $x_1^4+x_2^4+x_3^4=y_1^4+y_2^4+y_3^4$ in terms of binary quadratic forms with integral coefficients; e.g.,

$x_1=(4, -46, 202)$, $x_2=(8, -74, 206)$, $x_3=(15, -24, 69)$, $y_1=(10, 2, 28)$, $y_2=(14, -26, 32)$, $y_3=(9, -72, 243)$, where

$$(\lambda, \mu, \nu)=\lambda p^2+\mu pq+\nu q^2.$$

He also gives the solution,

$$x_1=128p^9+pq^8,\ x_2=64p^8q-12p^4q^5-q^9,\ x_3=0,$$

$$y_1=3pq^8,\ y_2=128p^9-2pq^8,\ y_3=64p^8q+12p^4q^5-q^9.$$

K. Mahler (Manchester)

836:

Smirnov, G. P. Theory of Hermitian forms over an arbitrary field with involution. Vestnik Leningrad. Univ. Ser. Mat. Fiz. Him. **9** (1954), no. 5, 41–52. (Russian)

837:

Fišman, K. M. The geometry of a binary form of fourth order. Černivec. Derž. Univ. Nauk. Zap. Ser. Fiz.-Mat. **19** (1956), no. 4, 75–82. (Ukrainian. Russian summary)

838:

Apostol, T. M.; and Sklar, Abe. The approximate functional equation of Hecke's Dirichlet series. Trans. Amer. Math. Soc. **86** (1957), 446–462.

Die approximative Funktionalgleichung der Riemannschen Zetafunktion von Hardy und Littlewood [Proc. London Math. Soc. (2) **21** (1922), 39–74] ist von verschiedenen Autoren auf Klassen weiterer spezieller Dirichletreihen ausgedehnt worden. Die Verfasser untersuchen dieses Problem für Heckes Dirichletreihen $\phi(s)$ der Signatur $(\lambda, \kappa, \gamma)$. Ausgehend von einer Formel von Apostol [Duke Math. J. **18** (1951), 517–525; MR **13**, 15] wird eine Identität hergeleitet, aus der dann mittels Hilfssätzen von Hardy und Littlewood eine einfachste und dann die allgemeine Form einer approximativen Funktionalgleichung für diese Dirichletreihen bewiesen werden. Der Spezialfall $\phi(s)=\zeta(2s)$ mit der Signatur $(2, \frac{1}{2}, 1)$ ergibt die erste Hardy-Littlewoodsche Form [Math. Z. **10** (1921), 283–317], d.h. einen zusätzlichen Logarithmusfaktor im Fehlerglied. Als Anwendung wird gezeigt, daß für die Abszisse σ_a der absoluten Konvergenz der Reihe von $\phi(s)$ stets $\sigma_a \leq \frac{1}{2}\kappa+\frac{1}{4}$ gilt [vgl. A. Z. Val'fiš, Soobšč. Akad. Nauk Gruzin. SSR. **16** (1955), 497–502; MR **17**, 349].

H.-E. Richert (Göttingen)

839:

Ignat'eva, R. P. The number of non-negative integral solutions of the equations $x+2y+3z+4u=m$, $x+2y+3z+4u+5v=m$ (a new method of solution). Kabardin. Gos. Ped. Inst. Uč. Zap. **8** (1955), 45–52. (Russian)

840:

Ignat'eva, R. P. The number of non-negative integral solutions of the equation $x+2y+3z+4u+5v+6w=m$. Kabardin. Gos. Ped. Inst. Uč. Zap. **8** (1955), 53–59. (Russian)

841:

Lomsadze, G. A. Representation of numbers as sums of generalized polygonal numbers. II. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **24** (1957), 3–33. (Russian)

842:

Mitsui, Takayoshi. On the partitions of a number into the powers of prime numbers. J. Math. Soc. Japan **9** (1957), 428–447.

The author develops an asymptotic formula for the number $T(n, m; k)$ of partitions of n into kth powers of prime numbers not exceeding m, where $m \leq n^{1/k}$, $k \geq 1$. He shows that $T(n, m; k) \sim (2\pi A_2)^{-\frac{1}{2}} \exp(n\alpha+A_1)$ as $n, m \to \infty$, where α is the root of the equation $n=\sum_{p \leq m} p^k(e^{\alpha p^k}-1)^{-1}$ and A_1, A_2 are positive constants defined by similar sums over the primes not exceeding m ($n\alpha \to \infty$ as $m, n \to \infty$). He writes

$$T(n, m; k)=\exp(n\alpha)\int_{-\frac{1}{2}}^{\frac{1}{2}} G(\exp(-\alpha+2\pi i\theta))\exp(-2\pi i n\theta)\,d\theta$$

where G is the generating function $G(w)=\prod_{p \leq m}(1-w^{p^k})^{-1}$ of $T(n, m; k)$. The author then follows a paper of Szekeres [Quart. J. Math. Oxford Ser. (2) **4** (1953), 96–111; MR **15**, 201] and shows that, with the above choice of α, the principal contribution to the integral comes from the neighborhood of $\theta=0$. For the difficult estimation of the error term he relies partly on methods and results of Vinogradov and Hua.

Some special cases are discussed in the last section. For example, if $m \leq n^{1/(k+1)}$, it is shown that $n\alpha \sim (m/\log m)$, $A_1 \sim (m/\log m)\log(nm^{-k-1}\log m)$, and $A_2 \sim n^2 m^{-1}\log m$. These results generalize those of Haselgrove and Temperley [Proc. Cambridge Philos. Soc. **50** (1954), 225–241; MR **16**, 17], who, as a special case of their theorems, obtained an asymptotic formula for $T(n, m; 1)$. They used a different method.

J. Lehner (East Lansing, Mich.)

843:

Gupta, Hansraj. Certain averages connected with partitions. Res. Bull. Panjab Univ. no. **124** (1957), 427–430.

Let m run over each of the $kp(n, k)$ parts occurring in the $p(n, k)$ partitions of n into exactly k parts, and define

$$S_r(n, k)=\sum_m m^r,$$

$$A_r(n, k)=S_r(n, k)/kp(n, k).$$

It is shown that for large n and a fixed k,

$$A_r(n, k)=\binom{r+k-1}{r}^{-1} n^r+O(n^{r-1}).$$

For $k=1, 2, 3$ exact formulas for $A_2(n, k)$ are given; for example, $A_2(n, 2)=\frac{1}{2}n(2n-1)$, n odd. The proofs depend on the facts that, for a fixed k and large n, $p(n, k)=n^{k-1}/k!(k-1)!+O(n^{k-2})$; and that

$$S_r(n, k)=\sum_{t=1}^{n}\sum_{i=1}^{j_t} p(n-it, k-i)t^r,$$

where $j_t=\min([n/t], k)$. *M. Newman* (Washington, D.C.)

844:

Dubois, D. W.; and Steger, A. A note on division algorithms in imaginary quadratic number fields. Canad. J. Math. **10** (1958), 285–286.

A function $g(x)$ is called a Euclidean function for the domain E if it is defined for every non-zero element x in E and if, for every x and y in E such that x does not divide y, there exists a q in E such that $g(y-qx)<g(x)$. It is well known that in certain quadratic fields the norm is a Euclidean function; such a field is necessarily a principal ideal field. It is also known that quadratic principal ideal fields exist for which the norm is not a Euclidean function [see, e.g., G. H. Hardy and E. M. Wright, An introduction to the theory of numbers, Clarendon Press, Oxford, 1954; MR **16**, 673; pp. 212–215]. The authors prove that in an imaginary quadratic field no other function than the norm can be a Euclidean function. The proof, which is very simple, depends on the fact that the number of units in such a field is finite,

and cannot therefore be extended to real quadratic or other fields. *H. W. Brinkmann* (Swarthmore, Pa.)

845:

Taussky, Olga. On matrix classes corresponding to an ideal and its inverse. Illinois J. Math. **1** (1957), 108–113.

It has been shown by the author that there is a one-to-one correspondence between classes of matrices with integral elements and determinant ± 1, and ideal classes in the ring R generated by a root of an irreducible algebraic equation $f(x)=0$ with integral coefficients [Canadian J. Math. **1** (1949), 300–302; MR **11**, 3]. In the present paper it is proved that transposed matrices correspond to inverse ideal classes; this fact had been proved by the author in an earlier paper in case $f(x)$ is quadratic [Pacific J. Math. **1** (1951), 127–132; MR **13**, 201]. It is also shown that the matrix classes defined by unimodular matrices coincide with the classes of determinant ± 1 if and only if R contains a unit with norm -1. Finally, it is shown, in case $f(x)$ is quadratic, that a matrix class corresponding to an ideal class of order 2 will contain a symmetric matrix if there is a unit of norm -1.

H. W. Brinkmann (Swarthmore, Pa.)

846:

Arima, Satoshi. Generation of algebraic function fields by their values of period two. Sûgaku **9** (1957/58), 11–13. (Japanese)

Let K be an algebraic function field of transcendental degree one over the complex number field C. Let G be the genus of K. Then there exist exactly $N=2^{2G}$ divisor classes $\{\bar{E}_i\}$ of degree zero with period two. Let E_i be arbitrarily chosen from $\bar{E}_i$ and let $E_i^2=(x_i)$ $(i=1, 2, \cdots, N)$. The author proves the following theorem. (i) Let K be neither elliptic nor hyperelliptic. Then $K=C(x_1, \cdots, x_N)$ holds. (ii) If K is hyperelliptic with genus $G\geqq 3$, then either $K=C(x_1, \cdots, x_N)$ holds or $k=C(x_1, \cdots, x_N)$ is rational and $[K:k]=2$. *Y. Kawada* (Tokyo)

847:

Lamprecht, Erich. Restabbildungen von Divisoren. Arch. Math. **8** (1957), 255–264.

Suppose that A is a field of algebraic functions of one variable with the precise field of coefficients K. The author extends and clarifies in this note some results of M. Deuring [Math. Z. **47** (1942), 643–654; MR **7**, 362] on the reduction of A/K modulo prime divisors $\mathfrak{p}$ of K, e.g., the behaviour of modules of multiples of a divisor of A, the product of divisors, etc., upon such reduction. His methods of proof and approach to the whole problem are those of Deuring [loc. cit.] and his recent note [same Arch. **8** (1957), 30–38; MR **19**, 396]. For a setting of these results in a more general context, see the recent work of Shimura. *O. F. G. Schilling* (Chicago, Ill.)

848:

Coles, W. J. On a theorem of van der Corput on uniform distribution. Proc. Cambridge Philos. Soc. **53** (1957), 781–789.

Van der Corput proved [Acta Math. **56** (1931), 373–456; **59** (1932), 209–328] that a sequence of points (α_n, β_n) $(1\leqq n<\infty)$ in the plane is uniformly distributed mod 1 if and only if for all pairs of integers u, v excluding $u=v=0$ the one-dimensional sequence $u\alpha_n+v\beta_n$ $(1\leqq n<\infty)$ is uniformly distributed mod 1. The author gives a quantitative form to this qualitative criterion. Denote by $F^{(N)}(x_0, x_1; y_0, y_1)$ the number of points (α_n, β_n) $(1\leqq n\leqq N)$ satisfying

$$x_0\leqq\alpha_n\leqq x_1 \text{ (mod 1)}, \quad y_0\leqq\beta_n\leqq y_1 \text{ (mod 1)}.$$

Put

$$D^{(N)}=\max \frac{1}{N} |F^N(x_0, x_1; y_0, y_1)-N(x_1-x_0)(y_1-y_0)|.$$

$D^{(N)}$ is said to be the discrepancy of the sequence (α_n, β_n) $(1\leqq n\leqq N)$. Denote by $D_{u,v}^{(N)}$ the (one-dimensional) discrepancy of $u\alpha_n+v\beta_n$ $(1\leqq n\leqq N)$. The author proves that there is an absolute constant C such that for every $\varepsilon>0$

$$D^{(N)}\leqq\varepsilon+C(D_{0,1}^{(N)}+D_{1,0}^{(N)}+\sum_{\substack{(u,v)=1\\ u>0,\ v>0}} f_{uv}(\varepsilon)D_{uv}^{(N)}),$$

where $f_{u,v}(\varepsilon)=\min(|uv|^{-1}, |\varepsilon uv|^{-2})$.

P. Erdős (Toronto, Ont.)

849:

Obrechkoff, N. Sur une question de l'approximation diophantique des formes linéaires. C. R. Acad. Bulgare Sci. **9** (1956), no. 4, 1–4. (Russian summary)

A proof of the following theorem: "Let α, β, and $r\geqq 1$ be arbitrary real numbers. There exist integers x, y, z not all zero such that

$$|\alpha x+\beta y+z|<\frac{\theta(1+\alpha^2+\beta^2)^{\frac{1}{2}}}{r^2}, \quad \max(|x|, |y|, |z|)\leqq r$$

if $\theta\geqq 1$; but this is not always true if $\theta<1$." This improves a much weaker result by E. Borel [J. Math. Pures Appl. (5) **9** (1903), 329–375]. *K. Mahler* (Manchester)

850:

Djerasimović, Božidar. Über die periodische Kettenbrüche. Bull. Soc. Math. Phys. Serbie **8** (1956), 137–146. (Serbo-Croatian. German summary)

Usant ses notations symboliques relatives à certaines opérations parmi les complexes ordonnés des nombres entiers [Math. Z. **62** (1955), 320–329; **66** (1956), 228–239; MR **17**, 255; **18**, 635], l'auteur exprime, au moyen de ces notations, une suite des relations qui existent entre la plus petite solution de l'équation de Pell $u^2-Dv^2=\pm 1$, et des fractions continues représentant deux nombres irrationnels quadratiques et conjugués mais arbitraires de discriminante D. *M. Tomić* (Belgrade)

COMMUTATIVE RINGS AND ALGEBRAS

See also 901, 904.

851:

Zierler, Neal. On the theorem of Gleason and Marsh. Proc. Amer. Math. Soc. **9** (1958), 236–237.

Let K be a finite field having q (a prime power) elements. Let f be irreducible over K. If f has roots in an extension field F then these roots all have the same order in the multiplicative group of F, and we may refer to this order as the order of f. Let α be the linear map of $K[x]$, considered as a vector space over K, into itself such that $\alpha(x^i)=x^{qi-1}$. Theorem: If f is irreducible in $K[x]$, then the degree of each irreducible factor of $\alpha(f)$ is equal to the order of f. This generalizes an observation of R. W. Marsh and the reviewer [A. A. Albert, Fundamental concepts of higher algebra, Univ. Chicago Press, 1958; p. 132] that if the roots of f are primitive then $\alpha(f)$ is irreducible. *A. M. Gleason* (Cambridge, Mass.)

852:
Hazanov, M. B. On the completeness of the real field. Kabardin. Gos. Ped. Inst. Uč. Zap. **8** (1955), 19–20. (Russian)

853:
Nagata, Masayoshi. Note on a paper of Lang concerning quasi algebraic closure. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **30** (1957), 237–241.

A field K is called C_i [strongly C_i] if every homogeneous form [polynomial without constant term] in K of n variables and of degree d with $n>d^i$ has a nontrivial zero in K. The paper proves: 1a) if $f_1, \cdots, f_r$ are r homogeneous forms of degree d in n common variables in a C_i field K, and if $n>rd^i$, then the forms have a common non-trivial zero in K; 1b) similarly with polynomials $f_1, \cdots, f_r$, without constant term, of degree at most d, in a strongly C_i field K; 2a) if L is an extension field of a C_i field K with $\dim_K L=r$, then L is C_{i+r}; 2b) similarly with "C_i" replaced by "strongly C_i". The results generalize those of Lang [Ann. of Math. **55** (1952), 373–390; MR **13**, 726] who proved 1a), 2a), 2b) under the assumption that K admitted a "normic form" of order i. The paper offers also several problems and remarks in the context; for example, it is observed that if L is the function field of a normal curve having a rational point over K and if L is C_i, then K is C_{i-1}. *T. Nakayama* (Nagoya)

854a:
Hattori, Akira. The solution of problem 6.1.13. Sûgaku **8** (1956/57), 207–208. (Japanese)

854b:
Hattori, Akira. On Prüfer rings. J. Math. Soc. Japan **9** (1957), 381–385.

[The second paper is the English translation of the first.] The author proves the following result, which answers the Problem 6.1.13 in Sûgaku **6** (1954/55), 47, proposed by the reviewer. Let Λ be an integral domain. Then the tensor product of any two torsion-free Λ-modules is also torsion-free if and only if Λ is a Prüfer ring.
M. Nagata (Cambridge, Mass.)

855:
Sakuma, Motoyoshi. On prime operations in the theory of ideals. J. Sci. Hiroshima Univ. Ser. A **20** (1956/57) 101–106.

Let o be an integral domain with quotient field K. A mapping, $\mathfrak{a}\to\mathfrak{a}'$, of the ideals of o into the ideals of o is called a prime operation if (1) $\mathfrak{a}\subseteq\mathfrak{a}'$, (2) $(\mathfrak{a}')'=\mathfrak{a}'$, (3) $\mathfrak{a}\subseteq\mathfrak{b}$ implies $\mathfrak{a}'\subseteq\mathfrak{b}'$, (4) $o'=o$, (5) $(a\mathfrak{a})'=(a)\mathfrak{a}'$ for any $a\in K$. Particular prime operations, $\mathfrak{a}\to\mathfrak{a}_a$ and $\mathfrak{a}\to\mathfrak{a}_b$ have been studied by Prüfer and Krull [W. Krull, Idealtheorie, Springer, Berlin, 1935]. The first of these is defined in terms of integral dependence and the second in terms of valuations. Krull has shown that when o is integrally closed in K, the equality $\mathfrak{a}_a=\mathfrak{a}_b$ holds for all finitely generated ideals $\mathfrak{a}$. The present paper gives an elegant proof (free of restrictions on o and of finiteness conditions) that the equality $\mathfrak{a}_a=\mathfrak{a}_b$ holds for all ideals. The identity of $\mathfrak{a}_b$ with the asymptotic closure $\mathfrak{a}_s$ that was proved by the reviewer in the case where o is noetherian is extended to the case of finitely generated ideals in arbitrary integral domains. *H. T. Muhly* (Iowa City, Iowa)

856:
Châtelet, A. Chaînes et décompositions d'idéaux semi-premiers. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1(49)** (1957), 5–10.

Various decomposition theorems are given for the elements of the lattice of semiprime ideals of a commutative ring with unity.
R. E. Johnson (Northampton, Mass.)

857:
Szász, Ferenc. Two ring-theoretic problems. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **6** (1956), 213–218. (Hungarian)

Hungarian version of the papers in Mat. Sb. N.S. **40(82)** (1956), 269–272; and Publ. Math. Debrecen **4** (1956), 237–238 [MR **18**, 788, 187]. *A. Kertész* (Debrecen)

858:
Jaffard, Paul. Dimension des anneaux de polynomes. Comportement asymptotique de la dimension. C. R. Acad. Sci. Paris **246** (1958), 3199–3201.

A étant un anneau commutatif muni d'un élément unité, la dimension de l'anneau des polynomes à n variables sur A est une fonction linéaire de n pour n assez grand. *Résumé de l'auteur*

859:
Jaffard, Paul. Dimension des anneaux de polynomes: La notion de dimension valuative. C. R. Acad. Sci. Paris **246** (1958), 3305–3307.

Définition et propriétés de la dimension valuative d'un anneau. Caractérisation d'un anneau A de dimension valuative finie par le comportement asymptotique de la dimension de l'anneau de polynomes à n variables sur A. *Résumé de l'auteur*

860:
Uchiyama, S. Sur un problème posé par M. Paul Turán. Acta Arith. **4** (1958), 240–246.

Let $Z(m, n)$ denote the number of sets $(Z_1, \cdots, Z_n)$ of complex numbers satisfying the system

$$S_{m+1}=S_{m+2}=\cdots=S_{m+n-1}=0,$$

where $S_k=Z_1^k+\cdots+Z_n^k$ and m is an integer $\geqq 0$. Two sets $(Z_1, \cdots, Z_n)$ and $(Z_1', \cdots, Z_n')$ are equivalent in $Z(m, n)$ if there exists a complex number $\lambda\neq 0$ such that

$$f(x; Z_1, \cdots, Z_n)=f(x; \lambda Z_1', \cdots, \lambda Z_n'),$$

where

$$f(x; Z_1, \cdots, Z_n)=\prod_{j=1}^{n}(x-Z_j).$$

Let $B(m, n)$ denote the number of classes of non-trivial sets in $Z(m, n)$ relative to this equivalence relation. The object of the paper is to determine the number $B(m, n)$. The following theorems are proved.

$$\text{(I)}\qquad \sum_{d\mid(m,n)} a(d)B\left(\frac{m}{d}, \frac{n}{d}\right)=\frac{(m+n-1)!}{m!n!},$$

where $a(1)=1$ and $a(d)=d^{-1}\prod_{p\mid d}(1-p)$.

$$\text{(II)}\qquad B(m, n)=B(n, m).$$

Clearly, II is an immediate consequence of I.
L. Carlitz (Durham, N.C.)

861:
Seres, Iván. Proof of a conjecture of I. Schur. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **6** (1956), 219–227. (Hungarian)

This is a Hungarian version of the paper in Acta Math.

Acad. Sci. Hungar. **7** (1956), 151–157 [MR **18**, 635]. *A. Kertész* (Debrecen)

862:

Nagata, Masayoshi. A general theory of algebraic geometry over Dedekind domains. II. Separably generated extensions and regular local rings. Amer. J. Math. **80** (1958), 382–420.

Ce travail fait suite à même J. **78** (1956), 78–116 [MR **18**, 600], et comprend les chap. III et IV de la série d'articles prévue par l'auteur. Il donne des résultats purement algébriques sur les anneaux locaux, qui sont nécessaires dans la suite.

Chapitre III – Préliminaires sur les produits tensoriels; produits tensoriels localisés et complétés d'anneaux semi-locaux. Etude des extensions séparables et régulières d'un anneau de Dedekind. L'auteur démontre la généralisation suivante du lemme de normalisation: soit A un anneau intègre de type fini sur un sous-anneau I et tel que A soit séparable sur I; il existe alors une base de transcendance séparante $(x_1, \cdots, x_d)$ et un élément non nul a de I tels que $A[a^{-1}]$ soit entier sur $I[a^{-1}, x_1, \cdots, x_d]$. De plus, si A est extension régulière de I et si I est un anneau de Dedekind, alors, pour tout idéal premier $\mathfrak{p}$ de I à l'exception d'un nombre fini d'entre eux, $\mathfrak{p}A$ est premier et $A/\mathfrak{p}A$ est extension régulière de $I/\mathfrak{p}$. Etude des produits tensoriels d'anneaux intégralement clos; l'auteur démontre en particulier que, si A et A' sont deux anneaux intégralement clos contenant un corps k, s'ils sont séparables sur k et si $A \otimes_k A'$ est intègre, alors $A \otimes_k A'$ est intégralement clos; la conclusion cesse d'être vraie si on suppose seulement que k est un anneau de valuation discrète.

Chapitre IV – Soit A un anneau local. L'auteur donne des exemples d'extensions entières monogènes $A[b]$ qui sont des anneaux locaux. Etude des transformés quadratiques des anneaux locaux intègres. Rappel des résultats de Serre sur la dimension homologique des anneaux locaux. L'auteur passe alors à l'étude des anneaux locaux factoriels, et montre que, si $\hat{A}$ est factoriel, il en est de même de A. Un anneau local régulier A est factoriel dans les cas suivants: $\dim(A) \leqq 2$, A est d'égales caractéristiques, A est non ramifié ou gentiment (="tamely") ramifié; l'auteur conjecture que c'est toujours vrai; il démontre par des méthodes homologiques le résultat suivant (dû à Zariski, mais non publié): si tout anneau local régulier de dimension 3 est factoriel, alors tout anneau local régulier est factoriel.

Un appendice démontre que tout anneau de fractions $A_{\mathfrak{p}}$ d'un anneau local régulier est régulier, ceci dans divers cas particuliers et par des méthodes élémentaires. Un autre généralise le théorème d'irréductibilité analytique aux localités. *P. Samuel* (Clermont-Ferrand)

ALGEBRAIC GEOMETRY

See also 862.

863:

Facciotti, Guido. Sulle reti di quadriche. Period. Mat. (4) **35** (1957), 94–103.

Sequel to the expository article in same Period. (4) **34** (1956), 284–293 [MR **19**, 63].

864:

Chisini, O. La superficie cubica. I. Period. Mat. (4) **35** (1957), 202–218.

In this first paper the author gives a clear and original treatment of the principal properties of cubic surfaces, presenting it as a preliminary introduction to the study of algebraic geometry.

M. Piazzolla-Beloch (Ferrara)

865:

Tallini, Giuseppe. Caratterizzazione grafica delle quadriche ellittiche negli spazi finiti. Rend. Mat. e Appl. (5) **16** (1957), 328–351.

The present paper [a continuation of G. Tallini, Ann. Mat. Pura Appl. (4) **42** (1956), 119–164; MR **19**, 55] gives a purely graphical characterization of all quadrics of elliptic type of a projective space, of any dimension $n>3$, over an arbitrary $GF(q)$ with $q>3$. *B. Segre* (Rome)

866:

Nakai, Yoshikazu. On the arithmetic normality of hyperplane sections of algebraic varieties. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **29** (1955), 159–163; corrections, **30** (1957), 201.

Let V^r be a projective variety defined over an algebraically closed field k having no singular subvariety of dimension $r-1$. Then the general hyperplane section C of V is arithmetically normal if and only if (i) V is arithmetically normal and (ii) $\dim H^1(V, L(C_n))=0$, for $n \geqq 0$. Moreover, in this case the general hypersurface section C_m of V is arithmetically normal for any integer $m>0$. (Here $L(C_m)$ denotes the stack of germs of rational functions on V such that $(f)+C_n>0$.) This theorem is proved with the help of abstract sheaf theory.

H. T. Muhly (Iowa City, Iowa)

867:

Shafarevich, I. R. Birational equivalence of elliptical curves. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 267–270. (Russian)

Elliptische Kurven über einem nicht algebraisch abgeschlossenen Körper k können möglicherweise erst in einer endlichen Erweiterung K von k auf die Weierstraßsche Normalgestalt $y^2=x^3+ax+b$ gebracht werden. 2 Kurven sind dann bei Gleichheit der absoluten Invariante $j=4a^3/(4a^3+27b^2)$ zueinander birational äquivalent. In der vorliegenden Note wird weiterhin gezeigt: Jede über k definierte elliptische Kurve, die zu einer über K erklärten Kurve in Weierstraßscher Normalgestalt äquivalent ist, definiert ein System von Automorfismen des algebraischen Funktionenkörpers $K(x, y)$ über K. Bei algebraischem k wird ferner bewiesen: Es gibt nur eine endliche Zahl nicht birational äquivalenter Kurven über k mit gleicher Invariante j und einem Primdivisor ersten Grades in einer gegebenen endlichen Erweiterung K/k. Ist k ein $\mathfrak{p}$-adischer Zahlkörper, der ja nur endlich viele Erweiterungen endlichen Grades besitzt, so läßt sich der letztgenannte Satz noch in folgender Weise verschärfen: Es gibt nur endlich viele birational verschiedene Kurven gegebenen Grades und gegebener Invariante j über dem $\mathfrak{p}$-adischen Körper k. *W. Burau* (Hamburg)

868:

Lascu, Alexandru T. I gruppi di omologia della varietà delle secanti di una curva algebrica iperspaziale. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 716–718.

Let Γ be a non-singular algebraic curve of degree n and genus p in complex projective space P^m of dimension m.

Let $P_{m,1}$ be the Grassmannian of lines in P^m, and let V be the subvariety of $P_{m,1}$ representing those lines which intersect Γ. The author computes the integral homology groups of V. The interesting ones are: $H_3(V)$, which is a direct sum of $2p$ copies of the integers and of $2p$ copies of the integers modulo n; $H_1(V)$, which is cyclic of order r, where r is an integer depending on the number of multisecants of Γ. *M. F. Atiyah* (Cambridge, England)

869:

Rosina, B. A. Sopra una corrispondenza $[1, n-1]$ che si incontra nella teoria diametrale e sulle proprietà che ne derivano per le curve algebriche piane. Ann. Univ. Ferrara. Sez. VII. (N.S.) **6** (1956–1957), 13–26.

Dans la présente Note l'a. établit une correspondance $[1, n-1]$ qui existe entre les diamètres d'une courbe plane algébrique d'ordre n et les correspondantes directions conjuguées et en déduit la propriété que les diamètres en général se disposent en groupes de $n-1$ diamètres parallèles. La direction des diamétres d'un même groupe est conjuguée à $n-1$ directions du plan, en général distinctes. — A l'aide de cette correspondance l'a. retrouve les principales propriétés diamétrales des courbes planes algébriques. — L'a. examine ensuite les cas où la correspondance $[1, n-1]$ devient singulière et donne l'équation de la courbe et les propriétés particulières dans les divers cas qui se présentent. *M. Piazzolla-Beloch* (Ferrara)

870:

Rosina, B. A. Costruzione per "piccola variazione" delle curve algebriche sghembe con l'ausilio delle loro secanti multiple. I, II. Ann. Univ. Ferrara. Sez. VII. (N.S.) **6** (1956–1957), 41–72, 73–120.

Dans ces deux Notes l'a. donne une méthode générale pour compléter la classification des courbes gauches algébriques de G. Halphen et M. Noether et l'applique pour construire effectivement par le procédé de "petite variation" les courbes d'ordre $n \leqq 7$ en même temps que leurs sécantes multiples. Il retrouve ainsi les familles et sous-familles de courbes indiquées dans la classification mentionnée et tous les cas particuliers de courbes (formant des sous-familles) qui présentent des plurisécantes de multiplicité $\geqq 5$ (en utilisant une formule de M. Piazzolla-Beloch). — L'a. se limite, comme Halphen et Noether, à considérer courbes sans singularités. La méthode qu'il donne est la suivante: Pour déterminer toutes les courbes gauches algébriques C_n^p d'ordre n et genre p (donnés), l'a. suppose connues toutes les espèces de courbes gauches algébriques C_{n-1} d'ordre $n-1$, ayant le genre compris entre zéro et le maximum possible pour l'ordre $n-1$. L'a. prend une de ces courbes d'ordre $n-1$, le genre de laquelle il indique par x. En associant à cette courbe une plurisécante convenable s_y $(1 \leqq y \leqq n-2)$ on obtient une courbe dégénérée $C_{n-1}{}^x + s_y$, d'ordre n et genre $x+y-1$. Pour obtenir une courbe dégénérée, d'ordre n et genre p, doit être $x+y-1=p$, équation indéterminée en x et y (considérées comme inconnues). Chaque résolution en valeurs entières et positives, écartées les solutions arithmétiques qui sont géométriquement impossibles, correspond à une courbe dégénérée d'ordre n et genre p, bien déterminée, et il n'y a pas d'autres. Cela posé, par chacune de ces courbes dégénérées on conduit deux surfaces algébriques d'ordre minimum, respectivement d'ordre l et d'ordre m, F_l et F_m (desquelles F_l irréductible), qui se couperont en $C_{n-1}{}^x + s_y$ et une courbe complémentaire (irréductible ou réductible) K_{lm-n}. Soit T_m une surface d'ordre m passant par la courbe K_{lm-n}. La surface du faisceau $F_m + \lambda T_m = 0$, qui s'obtient pour λ assez petite, coupe la surface F_l en dehors de la courbe K_{lm-n}, suivant une courbe gauche algébrique irréductible, d'ordre n et genre p, voisine de $C_{n-1}{}^x + s_y$. On construit ainsi toutes les espèces de courbes C_n^p existentes. L'a. a appliqué cette méthode pour construire effectivement les courbes d'ordre $n \leqq 7$. *M. Piazzolla-Beloch* (Ferrara)

871:

Chow, Wei-Liang. Remarks on my paper "The jacobian variety of an algebraic curve". Amer. J. Math. **80** (1958), 238–240.

In the paper quoted in the title [same J. **76** (1954), 453–476; MR **15**, 823] there is an unproved statement which is far from obvious, and another proof presents a gap in the sense that it holds only in special cases. That statement is now proved, and it is shown how the gap can be filled. *B. Segre* (Rome)

872:

Abhyankar, Shreeram. Coverings of algebraic curves. Amer. J. Math. **79** (1957), 825–856.

Exposé discursif des points où la théorie des ramifications des corps K de fonctions algébriques d'une variable en caractéristique $p \neq 0$ diffère de la théorie correspondante en caractéristique 0. Voici trois résultats frappants: (1) Pour tout K/k il existe x dans K tel que la valuation à l'infini de $k(x)/k$ soit la seule qui se ramifie dans K. (2) Il existe des revêtements non-ramifiés et non-résolubles de la droite affine. (3) La nombre d'extensions L de degré donné de K, telles que les valuations de K qui se ramifient dans L appartiennent à un ensemble fini donné, est fini.

Le résultat (1) est déduit de l'étude de la courbe $y^{p+1} + xy^p + 1 = 0$. L'auteur étudie ensuite les extensions du corps $k(x)$ des fonctions rationnelles à une variable qui sont ramifiées en 1, 2 ou 3 points; il en déduit (2). Puis il définit les "groupes fondamentaux algébriques" d'une variété V: étant donné un fermé D de V, on définit $\pi_A(V-D)$ [resp. $\pi_A'(V-D)$, resp. $\pi_A^*(V-D)$] comme étant le système projectif des groupes de Galois des extensions galoisiennes L de $k(V)$ ramifiées seulement dans D [resp. et de plus gentiment (="tamely") ramifiées, resp. et de plus telles que $[L:k(V)]$ ne soit pas multiple de p]; liens avec la théorie classique; conjectures sur la réduction modulo p.

L'auteur donne aussi une formule permettant de déduire la différente d'une extension L de K de celle de la plus petite extension galoisienne L' de K contenant L. Il démontre divers résultats relatifs au cas où le corps de base est celui des nombres complexes. *P. Samuel* (Clermont-Ferrand)

873:

Wallace, A. H. Algebraic approximation of curves. Canad. J. Math. **10** (1958), 242–278.

The title of this paper corresponds only to its first two portions, which develop the information used later to answer questions concerning the geometrical structure of real algebraic varieties, and thus it does not indicate the paper's full content. The first part of the paper deals with the approximation of curves formed by segments which are analytic arcs by curves formed by arcs which are parts of algebraic curves. This is done first for curves lying in the plane, then in n-space, then in a variety. Later, "in part III the sheets of a real algebraic variety are defined, namely, as sets maximal with respect to the property that any two points can be joined by an analytic arc in the set... Next, a local study of a real algebraic variety shows that it has locally the structure of a cell complex in

the sense of Whitehead." Finally, the author answers some questions posed earlier by the reviewer [Ann. of Math. (2) **56** (1952), 405–421; MR **14**, 403]. He shows that the number of sheets of a variety is finite, that they are closed sets, and that every point of the variety is contained in some proper sheet. A proper sheet is defined as one such that a neighborhood of some point of the sheet contains no points of the variety not in the sheet. This concept of "proper sheets" corresponds to the "sheets" of the reviewer's paper. A question raised by the author, concerning the cell decomposition of a variety, is that of a possible criterion for two cells with common frontier to lie in the same sheet. *John Nash* (Cambridge, Mass.)

874:

Maroni, A. Sulla definizione di varietà algebrica. Rend. Sem. Mat. Messina **1** (1955), 109–114.

875:

***Severi, F. Il teorema di Riemann-Roch per curve, superficie e varietà; questioni collegate.** Ergebnisse der Mathematik und ihrer Grenzgebiete. Neue Folge. Heft 17. Reihe: Algebraische Geometrie. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. viii+131 pp. DM 23.60.

In the present monograph several fundamental properties of geometry upon an algebraic variety V are centred around different theorems of the Riemann-Roch type. V is supposed throughout to be defined in the complex field, irreducible, of dimension $r \geqq 1$, and free from singularities or endowed with ordinary singularities. The work is divided into six sections, and contains a rich bibliography and interesting historical information, some of which is derived from the personal experience of the author.

§ 1 deals with the rational functions on V, and with the essential properties of the linear systems of hypersurfaces. The following § 2 concerns the case when V is a curve ($r=1$), and discusses the ordinary theorem of Riemann-Roch from the points of view of Riemann and of Brill and Noether; a theorem is then given assigning the dimension of a complete Abelian series of point-sets on a curve. § 3 begins by sketching the classical proofs by Castelnuovo, Severi and Enriques of the Riemann-Roch theorem for linear systems of curves on a surface, as well as the extensions — due to Severi — for linear systems defined by a reducible or virtual curve; then, several consequences are deduced and many properties concerning the super-abundance and the regularity of linear systems of curves are given. Next, in § 4, the different types of equivalences upon an algebraic variety are indicated, and a theorem of Riemann-Roch for complete irreducible algebraic systems of curves on a surface is obtained; moreover, B. Segre's principle of splitting is given, the phenomena of exorbitance and exuberance are illustrated, the two Picard varieties are introduced, and a necessary and sufficient condition that a virtual curve on a surface be algebraically equivalent to an effective curve is determined.

The varieties V of dimension $r \geqq 3$ are then studied in § 5, which is the culminating point of the monograph. It begins by summarizing the theory of the impure canonical system of type $r-1$ on V, and of the adjoint of a linear system of hypersurfaces. After some remarks on exceptional varieties of a birational transformation between varieties, the pure canonical system and the geometric genus of V are introduced as absolute invariants. Then the author's two definitions for the arithmetic genus are indicated, the effective and virtual characters of an effective or virtual subvariety of V (in particular, of a pure canonical hypersurface) are studied, several regularity theorems are given, and the equality of the two arithmetic genera is obtained. Next, the theorem of Riemann-Roch for linear systems of hypersurfaces is formulated both from the algebraic-geometric and from the transcendental point of view. An important connection between these two points of view is obtained by means of the relations among the irregularities of dimensions 1, 2, $\cdots$, r of V and the numbers of independent differential forms of the 1st kind of degrees 1, 2, $\cdots$, r on V. Finally, certain inequalities concerning irregularities and deficiencies are given, and a theorem on the regularity of the adjoint is obtained.

The last section gives some complementary notions and results on algebraic and rational equivalence, among which is the author's theorem affirming that every pure subvariety of V is rationally equivalent to a complete intersection of virtual hypersurfaces. Then the main properties of the rings of equivalence and the definitions of the canonical systems of diverse types 0, 1, $\cdots$, $r-2$ are recalled. Finally, the author's theorems generalizing those of Abel and of Riemann-Roch for point-sets on a surface are stated, and a number of unsolved questions are indicated. *B. Segre* (Rome)

876:

Nagata, Masayoshi. On the imbeddings of abstract surfaces in projective varieties. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **30** (1957), 231–235.

L'A. a déjà étudié autrefois le problème du plongement d'une variété abstracte dans une variété projective [mêmes Mem. **30** (1956), 71–82; MR **19**, 458]; la terminologie qu'il emploie est expliquée dans Amer. J. Math. **78** (1956), 78–116 [MR **18**, 600]. Ici il s'occupe du plongement d'une surface abstracte dans des variétés projectives; il démontre qu'il y a des surfaces normales complètes qui ne peuvent pas être plongées dans un espace projectif. *E. Togliatti* (Genova)

877:

Gröbner, W. Kontinuierliche Transformationsgruppen auf algebraischen Mannigfaltigkeiten. Monatsh. Math. **61** (1957), 209–224.

Einem homogenen Primideal $\mathfrak{p}$ der Dimension d des Polynomrings $\mathfrak{k}=K[x_0, \cdots, x_n]$ über dem Körper K kann man d modulo $\mathfrak{p}$ linear unabhängige Differentialoperatoren $D_i=\sum \vartheta_{ij}\partial/\partial x_j$ ($i=1, \cdots, d$) zuordnen, wobei die ϑ_{ij} für jedes i als Formen gleichen Grades angenommen werden können. Diese Operatoren sind dadurch gekennzeichnet, alle Formen aus $\mathfrak{p}$ zum Verschwinden zu bringen. Der Vollständigkeit halber wird zu ihnen noch der alle Formen annullierende Eulersche Operator $D_0=\sum x_i\partial/\partial x_i$ hinzugenommen. Diese D_i bilden dann einen $(d+1)$-dimensionalen Vektormodul über dem Restklassenkörper $\mathfrak{k}^*$ von $\mathfrak{k}$ modulo $\mathfrak{p}$. Nach Erklärung der üblichen Lieschen Kreuzmultiplikation erweist er sich als Liescher Ring, und es läßt sich stets eine solche Vektorbasis wählen, daß die Klammerausdrücke alle verschwinden. Nach den Grundgesetzen der Lieschen Theorie folgt dann, daß die $D_1, \cdots, D_d$ im kleinen eine d-gliedrige abelsche Gruppe auf der zu $\mathfrak{p}$ gehörigen algebraischen Mannigfaltigkeit M_d definieren. Für M_d bedeutet dies folgende wichtige Aussage: In der Umgebung eines regulären Punktes von M_d, d.h. eines solchen, wo die D_i stetig und linear unabhängig uber $\mathfrak{p}$ sind, läßt sich M_d durch ein System holomorpher Funktionen von d Parametern $t_1, \cdots, t_d$ darstellen. Die

Menge der nicht regulären Punkte ist von kleinerer Dimension als d, überall sonsthin lassen sich die genannten Funktionen unbeschränkt fortsetzen; sie liefern dabei eine Abbildung der geeignet zerschnittenen Mannigfaltigkeit M_d auf einen d-dimensionalen Parameterbereich der affinen Raumes der $t_1, \cdots, t_d$. Die d-gliedrige kontinuierliche Gruppe ist im t-Raum die Abelsche Translationsgruppe; sie ergibt bei Übertragung auf die M_d dort eine d-gliedrige Gruppe von im allgemeinen natürlich nicht birationalen Transformationen von M_d in sich, die in Bezug auf die regulären Punkte der Mannigfaltigkeit transitiv ist. Als Beispiel werden ebene Kurven 3. Grades herangezogen. Wie bei fast allen analytischen Untersuchungen zur algebraischen Geometrie wird der Grundkörper K als algebraisch abgeschlossen von der Charakteristik 0 angenommen. *W. Burau* (Hamburg)

878:

Matsusaka, T. Polarized varieties, fields of moduli and generalized Kummer varieties of polarized abelian varieties. Amer. J. Math. **80** (1958), 45–82.

The main result (section 3) is as follows. Let U be a projectively embeddable non-singular variety; then there is a maximal connected algebraic group G of automorphisms of U. Moreover, G is of finite index in the group G' of those automorphisms of U which, for a given projective embedding of U, carry a hyperplane section H into a cycle numerically equivalent to H. The paper also deals (in section 4) with the problem of defining a "field of moduli" for a complete non-singular polarized variety, defined up to isomorphisms; in the case of an elliptic curve, with the absolute invariant j, this field would be $k_0(j)$, where k_0 is the prime field. The existence of a field, with properties which make it reasonable to call it the "field of moduli" of the variety, is proved under mildly restrictive assumptions (always fulfilled in the case of characteristic 0). This may be considered as a first step towards the definition of a "variety of moduli"; as has been realized lately, it is no simple matter to construct or define a "variety of moduli", even for curves of given genus, so that the step carried out here is significant even in that case.

The proofs are based on earlier work of the author and on a number of highly technical lemmas contained in sections 1, 2, 3. One may quote, as particularly useful and fruitful, the author's earlier theorem [same J. **79** (1957), 53–66; MR **18**, 602] according to which, if $G_n(U)$ and $G_a(U)$ are the groups of divisors numerically and algebraically equivalent to 0 on the non-singular projective variety U, $G_n(U)/G_a(U)$ is a finite group. Unfortunately, the author's style, added to the intrinsic difficulties of the subject, makes it exceedingly hard to check the accuracy of many technical details which seem essential for the validity of the proofs; this is all the more to be regretted, since his results are so valuable and important. It is to be hoped that he, or someone else, will some day give a completely lucid and completely convincing exposition of these topics. *A. Weil* (Princeton, N.J.)

879:

Engel, Wolfgang. Über die Nullklassen in algebraischen Funktionenkörpern von zwei Veränderlichen. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **5** (1955/56), 905–908.

Let K be an algebraic function field of two variables over the field of complex numbers. To each normal model of K is attached a Picard variety V, consisting of the 0-divisor classes of the model; since V is in fact independent of the model chosen it may be called the Picard variety of K. The author proves the well-known classical statement that for each integer n the number of classes X in V with $X^n=1$ is equal to $n^{2\pi}$, where π is the dimension of V. The terminology is that of Jung's theory of algebraic functions of two variables [H. W. E. Jung, Einführung in die Theorie der algebraischen Funktionen zweier Veränderlicher, Akademie Verlag, Berlin, 1951; MR **13**, 680] besides this the proof uses only standard arguments [see, e.g., Conforto, Abelsche Funktionen und algebraische Geometrie, Springer, Berlin, 1956; MR **18**, 68; p. 234].
P. Roquette (Hamburg)

880:

Ono, Takashi. Sur une propriété arithmétique des groupes algébriques commutatifs. Bull. Soc. Math. France **85** (1957), 307–323.

Soient k un corps de nombres algébriques, $k_\mathfrak{p}$ le complété de k pour un diviseur premier $\mathfrak{p}$ de k, $\mathscr{A}$ (resp. $\mathscr{A}_\infty$) l'anneau des familles $(x_\mathfrak{p})$ avec $x_\mathfrak{p} \in k_\mathfrak{p}$ pour tout diviseur premier $\mathfrak{p}$ de k, et où $x_\mathfrak{p}$ est entier pour presque tout $\mathfrak{p}$ fini (resp. pour tout $\mathfrak{p}$ fini). On a $k \subset \mathscr{A}$ et $\mathscr{A}_\infty \subset \mathscr{A}$. Si alors G est un sous-groupe du groupe $\mathrm{Gl}(n, k)$ des matrices $n \times n$ inversibles à coefficients dans k, défini par des équations polynomiales $f_\alpha(\cdots, u_{ij}, \cdots)=0$, ces mêmes équations définissent des sous-groupes J et J_∞ de $\mathrm{Gl}(n, \mathscr{A})$ et $\mathrm{Gl}(n, \mathscr{A}_\infty)$ respectivement, et l'on a $G \subset J$ et $J_\infty \subset J$. Lorsque G est défini par la représentation régulière d'un surcorps K de k de degré n, J est le groupe des idèles du corps K; on sait alors que $J/G \cdot J_\infty$ est fini. De là et du th. de Dirichlet sur les unités, on déduit que si G est commutatif, et si la représentation linéaire de G dans k^n est simple, puis semi-simple, alors $J/G \cdot J_\infty$ est fini. Si G est commutatif et unipotent, il est isomorphe comme groupe algébrique à k^r pour un entier r, et de la compacité de $\mathscr{A}/k$, on déduit la finitude de $J/G \cdot J_\infty$. De là, on déduit que si le groupe G est commutatif, le groupe $J/G \cdot J_\infty$ est fini, en utilisant le th. de structure de Kolchin-Borel; c'est le résultat central de cet article. On notera que si G est le groupe d'une forme quadratique, la finitude du nombre des doubles classes de J modulo G et J_∞ équivaut à la finitude du nombre de classes dans un genre de formes quadratiques; cette dernière a été démontré par C. L. Siegel. Lorsque $G=\mathrm{Sl}(n, k)$, la théorie des diviseurs élémentaires sur un anneau Dedekind prouve que $J=G \cdot J_\infty$.
P. Cartier (Princeton, N.J.)

881:

Shafarevich, I. R. Exponents of elliptic curves. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 714–716. (Russian)

Unter dem Exponenten f einer algebraischen Kurve γ vom Geschlecht g über dem Körper k versteht der Verf. den kleinsten positiven Grad eines Divisors auf γ. Es gilt $f \mid 2g-2$, sodaß nur bei elliptischen Kurven ($g=1$) mehr als endlich viele Werte für f in Frage kommen. Verf. zeigt nun in der vorliegenden Note, daß es im Falle des rationalen Zahlkörpers R elliptische Kurven über R mit beliebig großem f gibt. Es wird hierzu vor allem die von A. Weil [Amer. J. Math. **77** (1955), 493–512; MR **17**, 533] eingeführte Gruppe $H(\omega)$ aller birational äquivalenten elliptischen Kurven mit gegebener Jacobischen Kurve ω herangezogen und gezeigt daß $H(\omega)$ über dem rationalen Zahlkörper R Elemente beliebig hoher Ordnung besitzt.
W. Burau (Hamburg)

882:

Turri, Tullio. Trasformazioni involutorie in S_r costruite a mezzo di ipersuperficie di ordine $r+1$. Rend. Sem. Fac. Sci. Univ. Cagliari **27** (1957), 1–11.

Estendendo e modificando opportunamente procedi-

menti usati in S_3, l'Autore costruisce sistemi lineari completi ∞^r di V_{r-1}^{r+1} algebriche dello spazio S_r, secantisi ad r ad r, fuori di una curva fissa, in una coppia di punti P, P'. Un siffatto sistema lineare genera pertanto una trasformazione birazionale involutoria dell' S_r.

D. Gallarati (Genova)

883:

Turri, Tullio. Le trasformazioni antibirazionali involutorie in un S_n. Rend. Sem. Fac. Sci. Univ. Cagliari **27** (1957), 12–20.

L'Autore dimostra che ogni trasformazione antibirazionale involutoria in un S_n reale, birazionalmente distinta dal coniugio, è birazionalmente identica al prodotto del coniugio in S_n per una trasformazione birazionale involutoria reale, distinta da un'involuzione proiettiva reale con spazi di punti uniti reali. La nozione di trasformazione antibirazionale involutoria devesi ad A. Comessatti, il quale per primo ha introdotto siffatte trasformazioni nello studio delle superficie razionali reali. Nella Nota presente l'Autore fa alcune osservazioni di carattere critico sulle ricerche di A. Comessatti.

D. Gallarati (Genova)

884:

Wylie, C. R., Jr. Line involutions in S_3 in which every line meets its image. Univ. Nac. Tucumán. Rev. Ser. A. **11** (1957), 32–40.

In previous papers the author has examined the general properties of involutions in S_3 in which a general line does not meet its image; in the present note the author studies in particular the involutions in which every line meets its image, working also here on the nonsingular V_4^2 in S_5 into whose points the lines of S_3 are mapped by the well-known procedure of interpreting the Plücker coordinates of a line in S_3 as point coordinates in S_5. In this way the author enumerates the quadratic line involutions and gives some examples of cubic involutions of the type considered.

M. Piazzolla-Beloch (Ferrara)

LINEAR ALGEBRA

See 1349, 1389.

ASSOCIATIVE RINGS AND ALGEBRAS

885:

Szász, F. A. Ringe, deren von Null verschiedene endlich erzeugbare Unterringe untereinander isomorph sind. Rend. Circ. Mat. Palermo (2) **6** (1957), 114–116.

The class of rings described in the title is enumerated and found to comprise, in addition to the eligible zero rings, only the finite prime fields. The author also lists without proof the members of the class of rings defined by applying the isomorphism condition only to proper subrings.

W. G. Lister (Oyster Bay, N.Y.)

886:

Jans, J. P. The representation type of algebras and subalgebras. Canad. J. Math. **10** (1958), 39–44.

For a finite algebra Γ with identity over a field K, the number of inequivalent indecomposable Γ-modules of degree d over K is denoted by $g_\Gamma(d)$. Γ is said to be of finite [resp. bounded; or unbounded; or strongly unbounded] representation type when $\sum_d g_\Gamma(d)<\infty$ [resp. $g_\Gamma(d)=0$ for sufficiently large d; or there is arbitrarily large d with $g_\Gamma(d)\neq 0$; or $g_\Gamma(d)=\infty$ for an infinite number of d]. With a subalgebra Λ of Γ, the paper proves that if $\Gamma=\Lambda+C$ (Λ-two-sided direct sum) and Λ is of [strongly] unbounded type, then so is Γ; and that if Λ has an identity e and $\Gamma e\otimes_\Lambda e\Gamma=\Gamma+C$ (Γ-two-sided direct sum), then if Λ is of finite [bounded] type, so is Γ. Further, if Σ is a separable algebra over K then Γ and $\Gamma\otimes_K\Sigma$ are of the same representation type. Also it is proved that, with $\Omega\supset L$, $F\supset K$, $L\cap F=J$ all fields, $L\otimes_K F$ is of finite type if L, F are separable over J and J is obtained from K by a single adjunction, and $L\otimes_K F$ is of strongly unbounded type if J is not obtained from K by a single adjunction.

T. Nakayama (Nagoya)

887:

Conrad, Paul. Generalized semigroup rings. J. Indian Math. Soc. (N.S.) **21** (1957), 73–95 (1958).

There is a noticeable similarity between rings of matrices, of polynomials, of power-series, and of functions. This paper studies a general form of algebraic system which includes all these. It has some interesting properties of its own. In the course of the work some new theorems about semi-group rings are found.

H. A. Thurston (Vancouver, B.C.)

888:

Findlay, G. D.; and Lambek, J. A generalized ring of quotients. I, II. Canad. Math. Bull. **1** (1958), 77–85, 155–167.

The present papers are generalizations of the work of the reviewer [Proc. Amer. Math. Soc. **2** (1951), 891–895; MR **13**, 618] and Y. Utumi [Osaka Math. J. **8** (1956), 1–18; MR **18**, 7]. If R is a ring and M and N are (right) R-modules, let $K=K(N, M)$ be the set of all R-homomorphisms of submodules of N into M. Thus, each $\alpha\in K$ has associated with it a submodule N_α of N such that $\alpha\in \mathrm{Hom}_R(N_\alpha, M)$. If $\alpha, \beta\in K$, then $\alpha\leqq\beta$ if $N_\alpha\subset N_\beta$ and $\alpha x=\beta x$ for every $x\in N_\alpha$. For each $\alpha\in K$, there exists a maximal element $\beta\in K$ such that $\alpha\leqq\beta$. If β is unique, α is called a fractional homomorphism of N into M. Let $F(N, M)$ be the set of all fractional homomorphisms of N into M. If A is a submodule of N such that $\mathrm{Hom}_R(A, M)\subset F(N, M)$, then write $A\leqq N(M)$. The module M is called a rational extension of N if $N\leqq M(M)$. It is shown that each module N has a unique maximal rational extension N^*, and $N^*=F(R+Z, N)$ where Z is the ring of integers and $R+Z$ is the usual extension of R to a ring with unity. If R and S are rings such that $R\leqq S(S)$, then S is called a quotient ring of R. Each ring R is shown to have a unique maximal quotient ring R^*. In case R is left-faithful, $R^*=F(R, R)$ is Utumi's maximal quotient ring. Further results are obtained in case R is commutative, satisfies a descending chain condition, or is algebraic over a field.

R. E. Johnson (Northampton, Mass.)

889:

Zelinsky, Daniel. Non-commutative Galois theory. Translated by Nobuo Nobusawa. Sûgaku **8** (1956/57), 12–16. (Japanese)

Starting with Artin's formulation of Galois theory, the address surveys the non-commutative Galois theory as developed by Cartan [Ann. Sci. Ecole Norm. Sup. (3) **64** (1947), 59–77; MR **9**, 325], Jacobson [Amer. J. Math. **69** (1947), 27–36; MR **9**, 4], Nakayama [Trans. Amer. Math. Soc. **73** (1952), 276–292; MR **14**, 240], Rosenberg and the author [ibid. **79** (1955), 429–452; MR **17**, 231] and others. Emphasized are the use of the centralizer theorem

in the proof and the normality theorem as proved, in its precise form, by Rosenberg and the author.

T. Nakayama (Nagoya)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

See also 893, 898, 933.

890:

Gaĭnov, A. T. Identical relations for binary Lie rings. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 141–146. (Russian)

Mal'cev has defined [Mat. Sb. N.S. **36(78)** (1955), 569–576; MR **16**, 997] a binary Lie ring as a non-associative ring in which every pair of elements is contained in a Lie subring. The author shows that in the case of characteristic $\neq 2$, a binary Lie ring can be characterised by the identical relations: $xy+yx=0$ and $[(xy)y]x+[(yx)x]y=0$. This also leads to a simple proof of Albert's result [Trans. Amer. Math. Soc. **64** (1948), 552–593; MR **10**, 349] that a non-associative ring of characteristic zero is power-associative if and only if it satisfies the identical relations $(xx)x=x(xx)$ and $[(xx)x]x=(xx)(xx)$.

K. A. Hirsch (London)

891:

***Seligman, George B. A survey of Lie algebras of characteristic p.** Report of a conference on linear algebras, June, 1956, pp. 24–32. National Academy of Sciences-National Research Council, Washington, Publ. 502, v+60 pp. (1957).

The author gives a comprehensive list of known results on the structure of Lie algebras of characteristic $p>0$, with emphasis on the progress that has been made towards the determination of the simple Lie algebras. Proofs are omitted. There is a useful bibliography.

C. W. Curtis (Ithaca, N.Y.)

HOMOLOGICAL ALGEBRA

892:

***MacLane, Saunders. Homologie des anneaux et des modules.** Colloque de topologie algébrique, Louvain, 1956, pp. 55–80. Georges Thone, Liège; Masson & Cie, Paris, 1957. 375 fr. belges; 3000 fr. français.

The author points out that the existing homology theories for various algebraic structures all started with the study of the extensions of these structures. In particular, he shows that extensions of rings and of modules are completely determined by giving a certain set of functions satisfying certain identities. Then he poses two natural problems: (I) to construct a complex $R(\Lambda)$ for each ring Λ so that its 2-cocycles are precisely those functions one meets in a ring extension problem and satisfying the same identities, and (II) to construct for each Λ-module C a complex $M_\Lambda(C)$ such that its 1-cocycles are the functions one meets in a module extension problem and satisfying the same identities. The paper is devoted to the solution of these two problems.

The first construction studied is a generalization of the bar construction of Eilenberg and MacLane [Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 657–663; MR **12**, 520]. The definition of the construction is made over two abelian categories, thereby giving a universal method of obtaining the standard constructions of homological algebra.

The second construction is the cubical one. For each abelian group C, one obtains a complex $Q(C)$ together with an augmentation $\eta_Q: Q(C)\to C$. The main theorem about this construction is that if $0\to A\to E\to C\to 0$ is an exact sequence of abelian groups, then the induced map $Q(E)/Q(A)\to Q(C)$ is a chain equivalence. The major step of the proof of this theorem is the construction of homotopies by successive approximations.

After obtaining these two constructions, the complexes $R(\Lambda)$ and $M_\Lambda(C)$ are defined by: $R(\Lambda)=B(Q(\Lambda), \eta_Q)$, where B is the bar construction applied to $Q(\Lambda)$ ($Q(\Lambda)$ is a differential graded ring); and $M_\Lambda(C)=Q(C)\otimes_{Q(\Lambda)}B(Q(\Lambda), \eta_Q)$. It is seen that the cubical construction is concerned with the additive and the bar construction with the multiplicative structure of the modules and rings.

D. A. Buchsbaum (Providence, R.I.)

893:

Dixmier, J. Homologie des anneaux de Lie. Ann. Sci. Ecole Norm. Sup. (3) **74** (1957), 25–83.

From the introduction: "Soit A un anneau de Lie. Soit Θ un A-module. Le groupe de cohomologie ordinaire $H^2(A, \Theta)$ sert à classer les extensions B de A par Θ qui possèdent la propriété suivante: le groupe abélien B est une extension inessentielle du groupe abélien A par le groupe abélien Θ. Nous nous proposons de définir de nouveaux groupes de cohomologie, que nous noterons $HJ^n(A, \Theta)$, tels que $HJ^2(A, \Theta)$ permette de classer toutes les extensions de A par Θ. Nous ne résoudrons d'ailleurs ce problème que si A est sans 2-torsion. Pour définir les $HJ^n(A, \Theta)$, nous construirons un complexe standard. Un complexe analogue, pour les anneaux associatifs, a été obtenu récemment par S. MacLane."

The construction of the standard complex is very intricate and cannot be sketched here. The standard complex is equipped with a multiplication. The analogous theory of MacLane, for associative rings, has meanwhile been published [$\neq$ 892 above].

G. P. Hochschild (Urbana, Ill.)

894:

Morita, Kiiti; Kawada, Yutaka; and Tachikawa, Hiroyuki. On injective modules. Math. Z. **68** (1957), 217–226.

Let A be a ring (with unit element) satisfying minimum condition. After some studies on projective and injective modules, which are closely related to some results in Eilenberg, Ann. of Math. (2) **64** (1956), 328–336 [MR **18**, 558] and Eckmann and Schopf, Arch. Math. **4** (1953), 75–78 [MR **15**, 5], the paper proves that an injective (left) A-module is indecomposable if and only if its largest completely reducible submodule is simple, that every injective A-module is a direct sum of indecomposable ones, and that if Q is an injective A-module whose largest completely reducible submodule is faithful for A/N, N being the radical of A, then for every submodule L_0 of an A-module L the annihilator in L of the annihilator in $\mathrm{Hom}_A(L, Q)$ of L_0 coincides with L_0. Also a proof is given to Ikeda's theorem [Osaka Math. J. **4** (1952), 203–209; MR **14**, 719; cf. Eilenberg and Nakayama, Nagoya Math. J. **9** (1955), 1–16; MR **17**, 453] on quasi-Frobenius rings. For closely related results cf. G. Azumaya [Amer. J. Math. **81** (1959), 249–278].

T. Nakayama (Nagoya)

895:

Tachikawa, Hiroyuki. Duality theorem of character modules for rings with minimum condition. Math. Z. **68** (1958), 479–487.

Let A be a ring (with unit element) satisfying the minimum condition for left ideals, and Q an injective left A-module whose largest completely reducible submodule is a finitely generated faithful A/N-module, N being the radical of A. Let B be the (inverse) A-endomorphism ring of Q. For a left A-module L [resp. right B-module R] the right B-module $\mathrm{Hom}_A(L, Q)$ [resp. left A-module $\mathrm{Hom}_B(R, Q)$] is denoted by $\mathrm{Char}_Q L$ [resp. $\mathrm{Char}_Q R$]. The main theorem of the paper states that for every finitely generated left A-module L, $\mathrm{Char}_Q[\mathrm{Char}_Q L] \approx L$ and there is a 1-1 duality by annihilation between submodules of L and $\mathrm{Char}_Q L$. If A is a finite algebra over a field P and $L[R]$ a left [right] A-module, the right [left] A-module $L^* = \mathrm{Hom}_P(L, P)$ [$R^* = \mathrm{Hom}_P(R, P)$] is called the dual module of $L[R]$. The dual module of the right A-module A can be chosen as Q and, with this choice of Q, $\mathrm{Char}_Q L \approx L^*$. For closely related results, cf. G. Azumaya [Amer. J. Math. **81** (1959), 249–278]. *T. Nakayama* (Nagoya)

896:

Rosenknop, I. Z. On the H. Cartan algebra of a polynomial ideal. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 1218–1221. (Russian)

In the polynomial ring $R = R(x_1, \cdots, x_m)$ over a field let $I = I(F_1, \cdots, F_n)$ be an ideal with generators $F_1, \cdots, F_n$. Let $\Lambda = \Lambda(z_1, \cdots, z_n)$ be the exterior algebra with generators $z_1, \cdots, z_n$. Put $l_i = \deg x_i$ and $k_j = \deg z_j$ and form the graded algebra $C = R \otimes \Lambda$ so that $F_1, \cdots, F_n$ are homogeneous with respective degrees $k_1, \cdots, k_n$. In an arbitrary Cartan algebra $C(F_1, \cdots, F_n)$ the following formula holds:

$$\sum_q (-1)^q H^{(q)}(t) = \{(1-t^{k_1}) \cdots (1-t^{k_n})\} / \{(1-t^{l_1}) \cdots (1-t^{l_m})\}.$$

Next let $M = (k_1 + \cdots + k_n) - (l_1 + \cdots + l_m)$. Suppose the generators of the ideal $I(\bar{F}_1, \cdots, \bar{F}_n)$ are obtained from n completely independent polynomials in n variables by setting $x_n = x_{n-1} = \cdots = x_{n-s+1} = 0$; then $H^{(q)}(t) = t^M H^{(q)}(1/t)$. *R. A. Good* (College Park, Md.)

GROUPS AND GENERALIZATIONS

See also 887, 934, 935, 999, 1237, 1238, 1239, 1240, 1266.

897:

Takeuchi, Kensuke. The word problem for free algebraic systems. Sûgaku **8** (1956/57), 218–229. (Japanese)

This is an expository paper concerning the word problem for free algebraic systems including semi-groups, semi-groups with cancellation and groups. Also the embedding problem of partial $\mathfrak{A}$-algebras into $\mathfrak{A}$-algebras is discussed. *Y. Kawada* (Tokyo)

898:

Kostrikin, A. I. On the connection between periodic groups and Lie rings. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 289–310. (Russian)

Let $B_{q,p}$ be the reduced free group with q generators and the identical relation $x^p = 1$. Let L be the free ring on q generators with integer coefficients. In L we consider two ideals. A is the well-known Magnus ideal [J. Reine Angew. Math. **182** (1940), 142–149; MR **2**, 214] and $\bar{L} = L/A$. I is defined as $I = pL + I'$, where I' is the ideal of L generated by all $(p-1)$-fold Lie commutators $[uv^{p-1}]$, $u, v \in L$, in the usual notation. It follows from results of Hall [Proc. London Math. Soc. (2) **36** (1933), 29–95], Zassenhaus [Abh. Math. Sem. Hansischen Univ. **13** (1939), 1–100] Sanov [Izv. Akad. Nauk SSSR. Ser. Mat. **16** (1952) 23–58; MR **13** 721] and Magnus [Ann. of Math. (2) **52** (1950) 111–126; MR **12**, 476] that $A \geqq I$. Let L, A and I be split into the direct sums of modules L_n, A_n, I_n of homogeneous Lie polynomials. The rank of L_n is given by Witt's formula. The connection with the restricted Burnside problem is the following. Let B_i be the terms of the lower central series of $B_{q,p}$, and B_∞ their intersection. Then $\bar{B}_{q,p} = B_{q,p}/B_\infty$ is the restricted Burnside group. Now, B_{n+1}/B_n can be regarded as isomorphic to $L_n - A_n$. If $\bar{B}_{q,p}$ is finite, then its order is the same as that of $\bar{L}$. Since $A \geqq I$, for an affirmative solution of the restricted Burnside problem it is sufficient to prove the nilpotence of the ring $L_1 = L/I$, the Lie ring of characteristic p satisfying the $(p-1)$th Engel condition $[uv^{p-1}] = 0$, $u, v \in L_1$. But it is by no means clear that the two problems, finiteness of $\bar{B}_{q,p}$ and nilpotence of L_1, are actually equivalent. Sanov [loc. cit.] has found that $A_n = I_n$ for $p \leqq n \leqq 2p-2$ (for $n < p$ it is very easy to establish), and has conjectured that $A = I$. In the present paper, the author achieves two things: firstly, he proves an inclusion in the opposite direction; namely $A \leqq pL + I_0' \cap L$, where I_0' is the rational closure of the ideal I'; and secondly, he proves that $I_n = A_n$ for $n = 2p-1$ and $2p$. These results are obtained by a very detailed study of the homogeneous constituents I_n, which is made in § 1 by means of a non-commutative differential operator which generalises the one that occurs in the Baker-Hausdorff formula. There is some connection between this work and that of Lyndon [Trans. Amer. Math. Soc. **77** (1954), 202–215; **78** (1955), 329–332; MR **16**, 218, 792], which is based on Fox's free differential calculus. The long proofs of the above main results are contained in § 2. Here, use is also made of Magnus' work [loc. cit.]. In § 3 the results are evaluated numerically in the case of $q = 2$ generators. It turns out that the ranks of I_{p+n} and A_{p+n} coincide for $p \geqq 7$ and $0 \leqq n \leqq 4$. The restricted Burnside group $\bar{B}_{2,5}$ is shown to be finite [see also Kostrikin, Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 233–244; MR **17**, 126] of order 5^α with $31 \leqq \alpha \leqq 34$, and of class c, with $10 \leqq c \leqq 12$. (The dimension of the 9th factor of the lower central series is 6, not 4, as stated by Lyndon [loc. cit.].) It also follows that for $p \geqq 5$, there exists a finite group with two generators and the identical relation $x^p = 1$ whose class of nilpotency is $2p$. (J. A. Green had obtained the class $2p-2$ [J. London Math. Soc. **27** (1952), 476–485; MR **14**, 350] and Meier-Wunderli the class $2p-1$ [Comment. Math. Helv. **30** (1956), 144–174; MR **18**, 376].)

K. A. Hirsch (London)

899:

Heller, I. On linear systems with integral valued solutions. Pacific J. Math. **7** (1957), 1351–1364.

In n-space, let S be a set of vectors $s_1, s_2, \cdots, s_r$ such that if $s_i = \sum c_k s_k$, where the s_k are linearly independent, then each $c_k = 1, -1$, or 0. These sets are connected with special systems of linear inequalities with integral-valued basic solutions. It is shown that $r \leqq n(n+1)$; equality holds only for the edges of a simplex. An S which is not a set of simplex edges is given for $n = 4$, $r = 18$: $\pm e_i$, $\pm(e_i + e_{i+1})$, $\pm \sum e_i$ (i mod 4). The same results were formulated in Bull. Amer. Math. Soc. **62** (1956), 552, as statements on sets of generators in a free Abelian group. *T. S. Motzkin* (Los Angeles, Calif.)

900:

Rühs, Fritz. Über die einfach ausgearteten Rédeischen schiefen Produkte. J. Reine Angew. Math. **198** (1957), 81–86.

The paper continues investigations by Rédei [same J. **188** (1950), 201–227; MR **14**, 13] and Kochendörffer, [ibid. **192** (1953), 96–101; MR **15**, 852], and uses the same terminology and notation. The skew product $G \circ \Gamma$ of groups G and Γ consists of the pairs (a, α) with $a \in G$, $\alpha \in \Gamma$ and the multiplication rule $(a, \alpha)(b, \beta) = (ab^{\alpha}\beta^{\alpha}, a^{b}\alpha^{b}\beta)$, where b^{α}, $\beta^{\alpha} \in G$, a^{b}, $\alpha^{\beta} \in \Gamma$ are suitable functions of the stated arguments. A skew product is simply degenerate if precisely one of the four relations $b^{\alpha}=b$, $\beta^{\alpha}=1$, $a^{b}=1$, $\alpha^{b}=\alpha$ holds identically. The author shows that there are essentially two distinct types of simply degenerate skew products. One of them, with $b^{\alpha}=b$, turns out to be a twice-repeated Schreier extension of certain subgroups, while the other, with $\beta^{\alpha}=1$, is a simple Schreier extension of certain subgroups, of which one is a so-called Zappa-Szép product. *K. A. Hirsch* (London)

901:

Dlab, Vlastimil. Die Endomorphismenringe abelscher Gruppen und die Darstellung von Ringen durch Matrizenringe. Czechoslovak Math. J. **7(82)** (1957), 485–523. (Russian summary)

The endomorphism ring $R(G)$ of an abelian group G has been determined for a variety of groups by Shoda, Baer, Shiffman, Kiškina and others [for references see A. G. Kurosh, The theory of groups, vol. 1, Chelsea, New York, 1955; MR **17**, 124; Sec. 21]. The present paper seeks a matrix representation of $R(G)$ for a general abelian group G. Let G^{*} be the minimal complete group containing G. If $R(G^{*}, G)=\{a; a \in R(G^{*}), Ga \subset G\}$ and $A=\{a; a \in R(G^{*}), Ga=0\}$, then $R(G) \cong R(G^{*}, G)/A$. Since G^{*} is a direct sum of Prüfer groups $G(p^{\infty})$ and torsionfree groups isomorphic to the additive group Q^{+} of the rational number field Q, endomorphisms of G^{*} will involve: p-adic integers from $R(G(p^{\infty}))$; rational numbers from $R(Q^{+})$; and p-adic numbers from Hom $(Q^{+}, G(p^{\infty}))$. It is proved that $R(G^{*})$ is isomorphic to a matrix ring (of infinite order in general) having blocks of elements of the three types described above. Hence $R(G)$ is isomorphic to a subring of this matrix ring modulo some ideal. The results of the paper are then used to represent rings in a similar way. *R. E. Johnson* (Northampton, Mass.)

902:

***Piccard, Sophie. Sur les bases des groupes d'ordre fini. Avec une préface de Arnaud Denjoy.** Mémoires de l'Université de Neuchâtel, Tome 25. Secrétariat de l'Université, Neuchâtel, 1957. xxiv+242 pp.

Das Werk stellt eine ausführliche Untersuchung über die Basen der endlichen Gruppen dar und bildet somit eine Fortsetzung der Untersuchungen der Verf. über die Basen der endlichen Gruppen [dieselben Mém. **19** (1946); Sur les bases du groupe symétrique, II, Vuibert, Paris, 1948; MR **8**, 13; **10**, 281]. Kap. 1 behandelt das Allgemeine über die Basen einer endlichen Gruppe, wobei auch die Erzeugendensysteme besprochen werden. In Kap. 2 werden diejenigen Gruppen untersucht, die von einem zusammenhängenden und primitiven System von Zyklen derselben Ordnung erzeugt werden. Unter diesem Gesichtspunkt werden Basen für die symmetrische und für die alternierende Gruppe angegeben. Sodann wendet sich Kap. 3 der Anzahlbestimmung der Basen und der Untersuchung spezieller alternierender Gruppen zu. Die imprimitiven Gruppen und die Anzahl ihrer Erzeugenden werden im Kap. 4 untersucht. Kap. 5: Die Systeme von Substitutionen, die eine reguläre Gruppe erzeugen. Die beiden Schlusskapitel endlich behandeln den Satz von Hölder über die Automorphismen der symmetrischen Gruppe und die Gruppen von Mathieu, Burnside und Klein.

J. J. Burckhardt (Zürich)

903:

Nakamura, Kirio. On finite groups of odd order. Sûgaku **9** (1957/58), 11. (Japanese)

The following theorem is proved. Let G be a finite group of odd order n such that n does not contain any factor p^{r}, $r \geq 3$. Then G' is abelian.

Y. Kawada (Tokyo)

904:

Huppert, Bertram. Zweifach transitive, auflösbare Permutationsgruppen. Math. Z. **68** (1957), 126–150.

It is known that the degree of a doubly transitive solvable permutation group is a prime power. Moreover, the group $S(p^{n})$ of all semilinear mappings $x \to ax^{p^{\nu}}+c$ over a Galois field $GF(p^{n})$ is an example of such a group for each prime power.

The author proves: Let G be a doubly transitive, solvable permutation group of degree p^{n}. Then, by a suitable designation of permutation symbols, $G \leqq S(p^{n})$, except for degree $p^{n}=3^{2}, 5^{2}, 7^{2}, 11^{2}, 23^{2}, 3^{4}$. For the proof, the author considers as symbols on which G acts the p^{n} vectors $(x_1, x_2, \cdots, x_n)$, where $x_i \in GF(p)$. Now letting $G^{(0)}$ designate the subgroup of G leaving $(0, 0, \cdots, 0)$ fixed, and expressing $G=G^{(0)} \cdot T$, where T is the uniquely determined minimal normal subgroup, the proof consists of determining all possibilities for $G^{(0)}$.

L. J. Paige (Los Angeles, Calif.)

905:

Curzio, Mario. Alcune osservazioni sul reticolo dei sottogruppi d'un gruppo finito. Ricerche Mat. **6** (1957), 96–110.

The first part of this paper is concerned with the lattices $\varphi(G)$ and $L_N(G)$ of all composition subgroups of G and all normal subgroups of G, respectively, G being a finite group. It is proved that $\varphi(G)$ is relatively complemented if and only if every composition subgroup is a direct product of simple groups, and that $L_N(G)$ is distributive if no two normal subgroups of G have equal orders. In the remainder of the paper the notions of upper and lower quasi distributivity are introduced and studied as applied to subgroup lattices. An element x of a lattice L is called upper quasi distributive (u.q.d.) if, for all $y, z \in L$ such that $x \cup y = x \cup z$, we have $x \cup (y \cap z) = (x \cup y) = (x \cup z)$. Lower quasi distributivity (l.q.d.) is defined dually. The main results are the following. A subgroup of a finite special group is u.q.d. as an element of the lattice of all subgroups if and only if there is no other subgroup of the same order. Every subgroup of a finite group is u.q.d. if and only if every subgroup is l.q.d., and this can happen if and only if the group is cyclic.

D. G. Higman (Ann Arbor, Mich.)

906:

Chang, Bomshik; Jennings, S. A.; and Ree, Rimhak. On certain pairs of matrices which generate free groups. Canad. J. Math. **10** (1958), 279–284.

The 2×2 matrices $A=I+\alpha e_{21}$, $B=I+\beta e_{12}$ generate a subgroup G of the 2×2 matrix group. Fuchs-Rabinowitsch proved [Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **10** (1940), 154–157; MR **2**, 215] that A, B generate a free group if $\alpha\beta$ is transcendental. Sanov proved [Dokl. Akad. Nauk SSSR (N.S.) **57** (1947), 657–

659; MR 9, 224] the same result if $\alpha=\beta=2$. Brenner extended this [C. R. Acad. Sci. Paris 241 (1955), 1689–1691; MR 17, 824] to the case $\alpha=\beta$, $|\alpha|\geqq 2$, and to certain other cases. The present article makes a further extension, namely to the case

$$|\alpha\beta|\geqq 2,\ |\alpha\beta-2|\geqq 2,\ |\alpha\beta+2|\geqq 2.$$

The first step is to show that the product $\alpha\beta$ determines G to within isomorphism. Related results proved are: (1) The algebraic numbers $\alpha\beta$ for which G is free are dense in the complex plane; (2) $\{I+\alpha e_{12},\ I+i\alpha e_{12},\ I+\beta e_{21},\ I+i\beta e_{21}\}$ is the free product of two free abelian groups if $|\alpha|\geqq 2$, $|\beta|\geqq 2$. The question whether the interval $-2\leqq\alpha\beta\leqq 2$ contains a subinterval Q such that G is free for $\alpha\beta\in Q$ is left open. *J. L. Brenner* (Palo Alto, Calif.)

907:

Yien, Sze-chien. A system of relations of unimodular matrices and automorphisms of unimodular group. Sci. Record (N.S.) **1** (1957), no. 1, 13–17.

This paper reports and sketches proofs of satisfactorily complete relations among systems of generators of the $n\times n$ unimodular group $\mathfrak{M}_n$ (with integral entries) and of the proper subgroup $\mathfrak{M}_n^+$. Absolute completeness is not claimed. The relations obtained are enough to show that any automorphism of $\mathfrak{M}_n^+$ is composed of $X\to PXP^{-1}$ and $X\to X'^{-1}$, $P\in\mathfrak{M}_n$. For an automorphism of $\mathfrak{M}_n$ (n even) the additional (outer) automorphism $X\to(\det X)\cdot X$ is needed. The computations cannot be conveniently abstracted. The related work is Hua and Reiner, Trans. Amer. Math. Soc. **71** (1951), 331–348; **65** (1949), 415–426 [MR **13**, 328; **10**, 684]. A simplified unpublished discussion of the generators of $\mathfrak{M}_n$ is attributed to Wan. *J. L. Brenner* (Palo Alto, Calif.)

908:

Suprunenko, D. A. On linear solvable groups. Mat. Sb. N.S. **41(83)** (1957), 317–332. (Russian)

The author continues his investigations into the structure of linear soluble groups over an arbitrary field P [see Dokl. Akad. Nauk SSSR **83** (1952), 183–186; MR **14**, 14]. In § 1 he gives, apart from some lemmas on irreducible imprimitive subgroups of $\mathrm{GL}(n, P)$, a simple proof of Clifford's theorem [Ann. of Math. (2) **38** (1937), 533–550] that the enveloping algebra of a normal subgroup of an irreducible primitive subgroup of $\mathrm{GL}(n, P)$ is simple over P. In § 2 he studies maximal irreducible primitive soluble subgroups of $\mathrm{GL}(n, P)$. Such subgroups need not exist. But if there is one, say Γ, then there is a normal series $\Gamma\geqq C\geqq A\geqq F\geqq 1$, in which F is a maximal abelian normal subgroup of Γ, C is the centraliser of F in Γ, A/F is maximal among the normal subgroups of Γ/F containing C/F. It turns out that F is the multiplicative group of a field Σ contained in the full linear algebra P_m, and that the degree m of Σ over P divides n. C/A is, of course, isomorphic to a subgroup of the group of automorphisms of A/F, and Γ/C is isomorphic to a subgroup of the automorphism group of Σ relative to P. The author shows that the index of F in Γ is finite; in fact, $(\Gamma:F)\leqq n^2(n^2-1)(n^2-2)(n^2-4)\cdots(n^2-2^\alpha)$, where α is maximal subject to the last parenthesis being positive. § 3 contains the main result of the paper; namely, that $(A:F)=n^2/m^2$ in the above notation. This result had been announced [loc. cit.] without a proof, but only for a perfect ground field P. As an application, the author classifies all the maximal irreducible soluble subgroups of $\mathrm{GL}(n, P)$ in the case when P is algebraically closed and n is squarefree. *K. A. Hirsch* (London)

909:

Wan, Cheh-hsian. On the automorphisms of linear groups over a non-commutative Euclidean ring of characteristic $\neq 2$. Sci. Record (N.S.) **1** (1957), no. 1, 5–8.

Let R be a non-commutative Euclidean ring of characteristic $\neq 2$; $\mathrm{GL}_n[R]$ is the group of invertible $n\times n$ matrices with entries from R; $\mathrm{SL}_n[R]$ is the group generated by the matrices $I+re_{ij}$ (transvections).

An outer automorphism of $\mathrm{SL}_n[R]$ ($n\geqq 3$) is represented either by $X\to X^\sigma$ or $X\to(S^\tau)^{-1}$, where σ is an automorphism, τ an antiautomorphism of R, or by transformation under $\mathrm{GL}_n[R]$, or by the composition of the latter with one of the former.

The automorphisms of $\mathrm{GL}_n[R]$ follow immediately; the Dieudonné determinant is involved in these.

Certain other principal ideal domains can be treated by the same method. However, when $n=2$ there is an example attributed to Hua of an automorphism $X\to AXA^{-1}$, which is not of the above types. In this example $R=\{a+b\sqrt{-5}, a, b \text{ integers}\}$; $A\notin\mathrm{GL}_2[R]$;

$$A=\begin{pmatrix} 2, & 1+\sqrt{-5} \\ 1-\sqrt{-5}, & 2 \end{pmatrix}.$$

The methods of the paper extend to symplectic groups. *J. L. Brenner* (Palo Alto, Calif.)

910:

Shepperd, J. A. H. Separation groups. Proc. London Math. Soc. (3) **7** (1957), 518–548.

A separation group G is a group for whose elements there is defined a quaternary relation $[a, b, c, d]$ ("a and c separate b and d") satisfying the following conditions. i) For any a, b, c, d, $[a, b, c, d]$ or $[a, c, d, b]$ or $[a, b, d, c]$. ii) $[a, b, c, d]$ and $[a, b, d, c]$ if and only if $a=b$ or $c=d$. iii) $[a, b, c, d]$ implies $[d, c, b, a]$. iv) $[a, b, c, d]$ implies $[b, c, d, a]$. v) $[a, b, c, d]$, $[a, e, b, c]$ and $b\neq c$ implies $[a, e, b, d]$. vi) $[a, b, c, d]$ implies $[gah, gbh, gch, gdh]$ for all $g, h\in G$. The author investigates separation groups and gives a complete description of them in terms of fully ordered and betweenness groups [see J. A. H. Shepperd, J. London Math. Soc. **31** (1956), 240–248; MR **18**, 61]. In every separation group G there exists a unique maximal normal fully ordered subgroup H whose order induces in it the separation relation; it consists of all elements h satisfying $[h^{-1}, 1, h, h^r]$ for all $r\geqq 2$. The author shows that (if $G\neq H$) there exist only the following four classes of separation groups: 1) G/H is the cycle of order 2, and G is the extension of a fully ordered abelian group by the automorphism mapping every element on its inverse. 2) G/H is the cycle of order 2, and G is a non-trivial infinite betweenness group. 3) G/H is the four-group, and the inner automorphisms of G preserve the order of H. 4) G/H is a separation group under the induced relation, separation-isomorphic to a subgroup of the rotation group of a circle, the inner automorphisms of G induce order automorphisms in H, and the elements of G can be cyclically ordered in a natural way; all cyclically ordered groups lead to separation groups of the fourth type. Except for the first case, all periodic elements of G must lie in the center of G. Various sets of elements of G are defined which are used to distinguish these cases. Also an interesting "logarithm" function is introduced to establish the fourth case. *A. Jaeger* (Cincinnati, Ohio)

911:

Podderyugin, V. D. Conditions for orderability of a group. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 199–208. (Russian)

E. P. Šimbireva has established [Mat. Sb. N.S. **20(62)**

(1947), 145–178; MR 8, 563] the following conditions for a group to be orderable: 1) Sufficient, but not necessary, is the existence of a central system with torsionfree factors; 2) necessary, but not sufficient, is the existence of a soluble normal system with torsionfree factors. It is plausible to look for necessary and sufficient conditions somewhere between 1) and 2) Such conditions have, in fact, been given independently by Iwasawa [J. Math. Soc. Japan 1 (1948), 1–9; MR 10, 428] and Mal'cev [Izv. Akad. Nauk SSSR. Ser. Mat. 13 (1949), 473–482; MR 11, 323], but both authors impose on the factors of the system conditions extraneous to group theory. The object of the present paper is to give a purely group-theoretical formulation of necessary and sufficient conditions.

Theorem: A group G can be ordered if and only if it has a soluble normal system with the following properties: the system consists of complete sets of conjugate subgroups in G; if A and B are neighbouring subgroups of the system, $A<B$, and if C is the derived group of the normaliser of A in G, then $[C, B]<A$; all the constituent subgroups of the system are strictly isolated in G. This latter condition means that for every $g \in G$, a product of conjugate elements of g can lie in one of the subgroups of the system only if all the factors do. The proof is rather closely related to Mal'cev's [loc. cit.]. As a corollary, the author finds that the torsionfree abelian groups of rank 1 can be characterised as the only groups capable of archimedean, but not of any non-archimedean order.

K. A. Hirsch (London)

912:

Ree, Rimhak. The existence of outer automorphisms of some groups. II. Proc. Amer. Math. Soc. 9 (1958), 105–109.

If G is a group of exponent p such that $p^2<|G|<\infty$, the author shows that $|G|$ divides $|A(G)|$, the order of the automorphism group of G. Let G be a linearly ordered, finitely generated, torsion-free nilpotent group. Then its group of order-preserving automorphisms is likewise finitely generated, torsion-free and nilpotent. Finally, he shows that if a group G has a normal subgroup of prime index p, and if this subgroup has a non-trivial center of exponent p and of order $<p^p$, then there exists an outer automorphism of G. Reference is made to an earlier paper, I of the same title [same Proc. 7 (1956), 962–964; MR 18, 640] and to a note by the same author [Trans. Roy. Soc. Canada. Sect. III. (3) 48 (1954), 39–42; MR 16, 792].

F. Haimo (St. Louis, Mo.)

913:

Herstein, I. N.; and Ruchte, M. F. Semi-automorphisms of groups. Proc. Amer. Math. Soc. 9 (1958), 145–150.

The authors define a mapping ϕ of a group G to be a semi-automorphism of G provided (i) ϕ is 1-1 upon G and (ii) $\phi(aba)=\phi(a)\phi(b)\phi(a)$ for all a, b in G. {Reviewer's remark: A mapping ϕ satisfies (i), (ii) if and only if $\phi(a)=\phi'(a)g$ for all a in G where ϕ' satisfies (i), (ii) and fixes the identity element, e, of G and where g is a center element such that $g^2=e$. Hence (following Dinkines in the paper cited below) we shall add the requirement (iii) ϕ fixes the identity element of G. This will simplify some of the statements.}

Dinkines [Proc. Amer. Math. Soc. 2 (1951), 478–486; MR 12, 801] proved that if G is a (finite or infinite) symmetric or alternating group, every semi-automorphism of G is an automorphism or anti-automorphism. She conjectured a like fact when G is a simple group. The present authors prove the following: (Theorem 2) Let G be a (finite or infinite) simple group containing an element of order four. Then every semi-automorphism of G is an automorphism or anti-automorphism. The authors also come close to replacing the "four" in Theorem 2 by "two". Dinkines' results follow quite easily, though her conjecture still remains open.

The methods of the paper have a considerable interest in their own right. Departing once more from the authors, let us call a semi-automorphism ψ of a group G "normal" if ψ commutes with every inner automorphism of G. Note that $\psi(a)a=a\psi(a)$ for every a in G and that ψ fixes the elements of order two. And recall that the only normal automorphism of a centerless group is the identity mapping I. With this background, the authors' key theorem takes a simple form: (Theorem 1) Let G be a group which is generated by its elements of order two and has a trivial center. Then to each semi-automorphism ϕ of G there corresponds a normal semi-automorphism ψ such that $\phi\psi^{-1}$ is an automorphism.

At this point the authors turn to the study of a normal semi-automorphism ψ of a simple non-abelian group with elements of order two. They wish to prove (a) $\psi=I$ or $-I$; and they are led to consider (b) $\psi^2=I$. They show that if (a) is false, then ψ coincides with I or $-I$ precisely on the elements of order two and the identity element. They show that (b) implies (a). They show that (b) holds if ψ^2 fixes an element of order greater than two. And, finally, they show that ψ^2 must fix an element of order four, if one exists. This proves Theorem 2.

R. H. Bruck (Madison, Wis.)

914:

Kargapolov, M. I. On conjugacy of Sylow p-subgroups of a locally normal group. Uspehi Mat. Nauk (N.S.) 12(1957), no. 4(76), 297–300. (Russian)

It is well known that in an infinite group with one finite class of conjugate Sylow p-subgroups all the Sylow p-subgroups are conjugate. The converse statement, that if all the Sylow p-subgroups are conjugate their number is finite, has been proved by Baer [Duke Math. J. 6 (1940), 598–614; MR 2, 2] in the case of countable locally finite and normal groups. In this note the author proves this result for arbitrary locally finite and normal groups.

K. A. Hirsch (London)

915:

*** Zappa, G. Questioni relative al reticolo dei sottogruppi di un gruppo: conoscenze attuali e problemi aperti.** Convegno italo-francese di algebra astratta, Padova, aprile, 1956, pp. 49–58. Edizioni Cremonese, Rome, 1957. vi+72 pp.

The author surveys the state of knowledge of the relationship between a group G and properties of its lattice $L(G)$ of subgroups, as of the date of the conference. The subject is treated under the following categories: characterization of classes of groups by properties of their lattices of subgroups; the problem of which groups H have $L(H)$ isomorphic to $L(G)$ for given G; the analogous problem for homomorphism; and the determination of special elements or categories of elements of $L(G)$, such as neutral or modular elements. For each category, the principal developments are summarized. This leads to an indication of open questions. Mostly these are natural analogs of settled questions. However, the most productive suggestion may be the general one of using the subgroup lattice characteristics — not only of $L(G)$ but also the lattice of normal subgroups and of groups appearing in composition series — as a basis for classifying large categories of groups.

As compared to the monograph of Michio Suzuki [Structure of a group and the structure of its lattice of subgroups, Springer, Berlin-Göttingen-Heidelberg, 1956; MR 18, 715] the present discussion is distinguished by: brevity (with resulting clarity in the main points, but omission of many other points, including precise statements where complex), omission of proofs, more explicit discussion of open questions, more emphasis on sublattices of $L(G)$ and less on detailed properties of groups, greater prominence of chronology, and a bibliography extending about two years later but somewhat less extensive for earlier work. *P. M. Whitman* (Baltimore, Md.)

916:

Sadovskiĭ, L. E. The lattice of subgroups of a nilpotent torsion-free group. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 201–204. (Russian)

Kontorovič and Plotkin have proved [Mat. Sb. N.S. **35(77)** (1954), 187–192; MR **16**, 440] that the lattice-isomorphic image of a locally nilpotent torsion-free group is also locally nilpotent and torsion-free. In this note the author gives an elementary proof of this fact. He also proves that a free nilpotent finitely generated group is completely determined by the lattice of its subgroups, i.e., is group-isomorphic to every lattice-isomorphic image. This result has also been obtained by Plotkin [Ukrain. Mat. Ž. **8** (1956), 325–329; MR **18**, 448]. The corresponding results for free abelian groups and free groups have been known for some time [Baer, Amer. J. Math. **61** (1939), 1–44; Sadovskiĭ, Dokl. Akad. Nauk SSSR **32** (1941), 171–174; MR **3**, 195].

K. A. Hirsch (London)

917:

Berlinkov, M. L. On the lattice of subgroups of a group with finite layers. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 267–271. (Russian)

Let S be a σ-lattice, i.e., one in which every countable subset has an inf and a sup. The lower and upper (0)-limits of a sequence (a_n) of elements of S are defined as $\underline{a}=\bigcup_{n=1}^{\infty}\bigcup_{k=n}^{\infty} a_k$, $\bar{a}=\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} a_k$. If $\underline{a}=\bar{a}$, the sequence is called (0)-convergent. A subset M of S is called (0)-compact if every sequence of its elements contains a (0)-convergent subsequence. By induction the concepts of (m)-convergence and (m)-compactness are defined: (a_n) is $(m+1)$-convergent if the segment $[\underline{a}, \bar{a}]$ is (m)-compact, and M is $(m+1)$-compact if every sequence of its elements contains an $(m+1)$-convergent subsequence. [Cf. 826.]

These concepts are used to give lattice-theoretical characterizations of groups with finite layers whose group-theoretical investigation is due to Černikov [Mat. Sb. N.S. **22(64)** (1948), 101–133; MR **9**, 566; **13**, 622]. The following conditions on the subgroup lattice $S(G)$ of a group G will play a rôle: (α) $S(G)$ satisfies the minimal condition; (β) $S(G)$ is m-compact for at least one m; (γ) every segment $[0, a]$ of $S(G)$ satisfying the minimal condition is m-compact for at least one m; (δ) if (a_n) is a sequence of $S(G)$ for which $[0, \bigcup_{n=1}^{\infty} a_n]$ does not satisfy the minimal condition, then there is a subsequence (b_n) with $b_n \leqq a_n$ such that the segment $[0, \bigcup_{n=1}^{\infty} b_n]$ is isomorphic to the complete lattice of all subsets of a countable set (it follows easily from results of Baer [Amer. J. Math. **61** (1939), 1–44] that a group has its subgroup lattice isomorphic to this lattice if and only if it is the direct product of an infinite number of cyclic groups with pairwise distinct prime orders); (ε) every segment $[0, a]$ of $S(G)$ satisfying the minimal condition is finite. The author proves the following theorems: 1. A group G is a Π_0-group with finite layers (where Π_0 denotes an arbitrary finite set of primes) if and only if its subgroup lattice satisfies (α) and (β). In particular: G is a Černikov p-group. (i.e., locally finite with minimal condition) if and only if $S(G)$ satisfies (β). 2. A group G has finite layers if and only if $S(G)$ satisfies (γ) and (δ). 3. A group G has finite layers and no infinite Sylow p-subgroups if and only if $S(G)$ satisfies (δ) and (ε). It follows that the properties under 2. and 3. are preserved under lattice isomorphisms. *K. A. Hirsch* (London)

918:

Scott, W. R. Solvable factorizable groups. Illinois J. Math. **1** (1957), 389–394.

Continuing investigations by Huppert and Itô [Math. Z. **59** (1953), 1–7; **61** (1954), 94–99; **64** (1956), 138–148; Acta Sci. Math. Szeged **14** (1951), 83–84; MR **15**, 197; **16**, 332; **17**, 940; **14**, 13], the author exhibits further cases in which a factorizable finite group $G=HK$ turns out to be soluble. They are: (1) H nilpotent and K abelian or hamiltonian; (2) H nilpotent of odd order, and K with a subgroup L of index 2 such that every subgroup of L is normal in K; (3) H cyclic, and K with a normal subgroup L of index 2 or 3 such that every subgroup of L is normal in K; (4) H and K both dicyclic or dihedral.

K. A. Hirsch (London)

919:

Kikodze, E. B. On complex commutators of elements of a group. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 301–303. (Russian)

Let $a_1, a_2, \cdots, a_n$ be elements of a group G. A complex commutator is denoted by $[a_1, a_2, \cdots, a_n]_{\alpha_n}$, where the symbol α_n refers to a certain distribution of brackets. This α_n can be described uniquely by a set of natural numbers $\alpha_n=(i_1, i_2, \cdots, i_{n-1})$, so that i_r is the index of that element which comes after the rth bracket [. G is called an α_n-group if $[a_1, a_2, \cdots, a_n]_{\alpha_n}=1$ for all choices of the $a_i \in G$. For example, a nilpotent group of class k is an α_{k+1}-group for $\alpha_{k+1}=(1, 1, \cdots, 1)$ and, in fact, for any distribution pattern. The author shows first that the groups which are α_n-groups for some n and some α_n are precisely the soluble groups. He then proves the converse of the above statement about nilpotent groups: If in a group G, a complex commutator $[a_1, a_2, \cdots, a_n]_{\alpha_n}$ is independent of the distribution α_n of brackets, so that $[a_1, a_2, \cdots, a_n]_{\alpha_n}=[a_1, a_2, \cdots, a_n]_{\beta_n}$ for all choices of the $a_i \in G$ and arbitrary distributions α_n and β_n, then G is nilpotent of class $\leqq n-1$. For $n=3$, this has been proved by F. W. Levi [J. Indian Math. Soc. (N.S.) **6** (1942), 87–97; MR **4**, 133]. *K. A. Hirsch* (London)

920:

Černikov, S. N. On groups with finite classes of conjugate elements. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 1177–1179. (Russian)

The author proves first that the class of groups with finite layers [see Černikov, Mat. Sb. N.S. **22(64)** (1948), 101–133; MR **9**, 566; **13**, 622] coincides with the class of locally finite and normal groups whose Sylow subgroups satisfy the minimal condition for abelian subgroups. In particular, a locally finite and normal group with minimal condition for abelian subgroups is either finite or a finite extension of a direct product of a finite number of quasicyclic groups. Next he shows that in an *FC*-group the factor group of the centre is periodic, and that the centre of a group is of finite index if and only if the centralizer of every abelian subgroup is of finite index.

Moreover, if in an infinite periodic group the centralizer of every abelian subgroup is of finite index, then the group is the direct product of an infinite abelian group and a finite group. Finally, if the centraliser of each abelian p-subgroup of a periodic group has finite index, then the factor group of the centre is a group with thin finite layers [for the definition, see loc. cit.].

K. A. Hirsch (London)

921:

Plotkin, B. I. On certain classes of infinite groups. Uspehi Mat. Nauk (N.S.) **13** (1958), no. 1(79), 189–192. (Russian)

The notion of a W-group [Trudy Moskov. Mat. Obšč. **6** (1957), 299–336; MR **19**, 529] is generalized to that of a $\overline{W}$-group. In a group $\mathfrak{G}$ the characteristic subgroup generated by all the completely reducible normal subgroups is denoted by $A(\mathfrak{G})$. A group $\mathfrak{G}$ is said to be F-semisimple if the centralizer of $A(\mathfrak{G})$ is the identity. A group is called a $\overline{W}$-group if it is the extension of a radical group by an F-semisimple group. A necessary and sufficient condition that a group be a $\overline{W}$-group is that it possess an ascending normal series each factor of which is either locally nilpotent or F-semisimple. A similar treatment for the class of WF-groups [Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 648–651; MR **17**, 1107] introduces the class of $\overline{WF}$-groups.

R. A. Good (College Park, Md.)

922:

Stein, Sherman K. On the foundations of quasigroups. Trans. Amer. Math. Soc. **85** (1957), 228–256.

Given a quasigroup Q, by applying a particular permutation σ to a, b, c in every relation $ab=c$ $(a, b, c \in Q)$ we obtain a quasigroup which the author denotes σQ and calls a conjugate of Q. A quasigroup thus has, including itself, six conjugates, one for each element σ of the symmetric group S_3. Any constraint (this term is defined precisely) satisfied by a quasigroup yields a conjugate constraint satisfied by a conjugate quasigroup. For example, one of the constraints conjugate to associativity is the identity $ab\cdot ac=bc$; hence, the theory of groups is equivalent to the theory of quasigroups satisfying this latter identity. Among further results in this context is the theorem that if all the conjugates of a group are groups, the group is a power of the group of order 2.

The left-distributive law $a\cdot bc=ab\cdot ac$ is an example of a constraint which is an identity possessing a conjugate constraint which is not an identity; its conjugate by either of the permutations (13), (132) is: $ab=cd$ implies $ac\cdot bd=ab(=cd)$. It is shown that there are left-distributive quasigroups of order n for and only for $n\equiv 0, 1, 3 \pmod 4$. Further theorems concern the automorphisms of left-distributive quasigroups. There is a brief section on quasirings: roughly, a quasiring is a collection of quasigroups on the same set which are left-distributive over one another.

The identity $ab\cdot cd=ac\cdot bd$, which under various names has attracted much attention, is interesting as being identical with its conjugates. The author calls it the medial law and studies at some length (i) constraints related to mediality, (ii) constraints equivalent to mediality, (iii) consequences of mediality. One result from each of these sections may be quoted. (i) A quasigroup satisfying the identities $ab\cdot ac=bc$, $ab\cdot b=a$ is medial; this is proved as the (23) conjugate of the fact that an abelian group is medial. (ii) An algebra A (i.e., a set closed under multiplication) is medial if and only if for any algebra B the product of any two homomorphisms of B into A (defined by taking the product of images in A) is again a homomorphism. (iii) If N is a subquasigroup of a medial quasigroup A, then the cosets $\{aN\}$ are a medial quasigroup under the law $aN\cdot bN=ab\cdot N$. Numerous examples of medial quasigroups are given, mostly geometrical, some original and some quoted.

Next (in the hope, his introduction suggests, of finding a counterexample to Euler's conjecture that there are no orthogonal quasigroups of order $4k+2$) the author gives various constraints such that if a quasigroup Q satisfies one of them, then either a quasigroup orthogonal to Q can be constructed or a pair of conjugates of Q will be orthogonal. Finally, conjugation is applied to varieties.

A most valuable feature of the paper is a chapter headed "Commentary" in which each section of the paper is placed in its context with references to the literature. This is supplemented by a bibliography of 41 items, and a "historical addendum" giving 7 more going back to Grassmann, Hankel and Schröder. There is also a list of "questions and problems".

I. M. H. Etherington (Edinburgh)

923:

Isbell, J. R. Some remarks concerning categories and subspaces. Canad. J. Math. **9** (1957), 563–577.

The author considers a category A and a subset I of A. He then looks for conditions under which A may be converted into a bicategory in the sense of MacLane [Bull. Amer. Math. Soc. **56** (1950), 485–516; MR **14**, 133] such that I becomes the set of inclusions of A. The author also is interested in the notion of an identification category (derived from a category with an equivalence relation), as well as a number of other rather disjoint questions.

S. Eilenberg (New York, N.Y.)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 877, 891.

924:

Cartier, P.; et Dixmier, J. Vecteurs analytiques dans les représentations de groupes de Lie. Amer. J. Math. **80** (1958), 131–145.

Let φ be a continuous representation of a Lie group G in a Banach space B. An element $b \in B$ is called an analytic vector for the representation φ if the map $x \to \varphi(x)b$ of G into B is analytic. Harish-Chandra has proved [Trans. Amer. Math. Soc. **75** (1953), 185–243; MR **15**, 100] that if G is connected and semisimple and φ is 'permissible', in the sense that it maps the elements of a certain discrete central subgroup of G onto scalar multiplications, then the analytic vectors are dense in B. Harish-Chandra's proof results from combining the case of a compact group with the case of a quasi-nilpotent group. The present authors make a precise analysis of the case of a solvable group, proving that, if G is connected and solvable and φ is any continuous representation of G in B, then the analytic vectors for φ are dense in B. Furthermore, they simplify and strengthen other parts of Harish-Chandra's proof to obtain the following general result.

If G is connected there is no loss of generality in assuming that G is simply connected. Then Iwasawa's decomposition theory shows that G has closed connected subgroups K and N, and a discrete central subgroup Z, such that $G=KN$, $K\cap N=(1)$, N is simply connected

and solvable, K contains Z, and K/Z is compact. The main result can now be stated as follows. The analytic vectors are dense in B in each of the following three cases: (1) G is connected and simply connected, and $\varphi(z)$ is a scalar multiplication in B, for every $z \in Z$; (2) G is connected and simply connected, and the set of operators $\varphi(z)$, with $z \in Z$, is bounded in the Banach space $L(B)$ of all continuous operators on B; (3) G is an arbitrary Lie group, and the set of all $\varphi(x)$, with $x \in G$, is bounded in $L(B)$. *G. P. Hochschild* (Berkeley, Calif.)

925:

Harish-Chandra. Spherical functions on a semisimple Lie group. I. Amer. J. Math. **80** (1958), 241–310.

Let G be a connected semisimple Lie group with a finite center. Let K be a maximal compact subgroup of G. The spherical functions (with respect to K) on G are the complex-valued functions which are invariant under right and left translation by members of K. It is known that there exist certain spherical functions, which are analytic on G, in terms of which a more or less arbitrary spherical function may be expanded in a Fourier-integral-like fashion. These are the elementary spherical functions. Moreover, if S denotes the space of all elementary spherical functions, one can find a measure μ in S for which an analogue of the Plancherel formula holds. The problem treated in this paper is that of explicitly determining the measure μ.

The elementary spherical functions turn out to be eigenfunctions of certain differential operators and this fact is exploited in the proof that a certain explicitly constructed family of them is exhaustive. The differential operators are those which Berezin calls "radial parts of Laplace operators" [Dokl. Akad. Nauk. SSSR (N.S.) **107** (1956), 9–12; **110** (1956), 897–899; MR **17**, 1109; **19**, 292]. By considering the asymptotic behaviour of these functions the author is led to a certain function c. S is closely related to a certain vector space E, and c defines a function on E which is analytic except on certain hyperplanes. The measure μ is obtained from Lebesgue measure in E by dividing by the square of the absolute value of c.

In the present paper the asymptotic behaviour of the members of S and the properties of the function c are investigated in some detail. However, the proof that the measure defined by c leads to a Plancherel formula is deferred to a subsequent article. The arguments are long and involved and make use of some unpublished results of Chevalley about Lie algebras.

The author points out that there is some overlap between a part of his work and the two notes of Berezin quoted above. Also some of his results were announced in a preliminary note [Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 408–409; MR **19**, 292.]

G. W. Mackey (Cambridge, Mass.)

926:

de Siebenthal, Jean. Sur les groupes de Lie compacts non connexes. Comment. Math. Helv. **31** (1956), 41–89.

Let G be a compact Lie group, G_0 its identity component. In Chapter I the author studies the group $A(G_0)$ of all automorphisms of G_0, which is a Lie group whose identity component is the group $I(G_0)$ of inner automorphisms of G_0. It is shown that $A(G_0)$ is a semi-direct product of $I(G_0)$ and a certain discrete group U. [An analogous theorem for semi-simple Lie algebras is due to Dynkin, Dokl. Akad. Nauk SSSR (N.S.) **76** (1951), 629–632; MR **12**, 585. See also Mostow, Mem. Amer. Math. Soc. no. 14, (1955), 31–54; MR **16**, 1087.] If G_0 is semi-simple, U is finite; it is determined explicitly for all simple compact groups. {On page 41, U is stated to be finite instead of discrete.} The group U is used to describe all group extensions E of simple groups G_0 which are natural in the sense that the centralizer of G_0 in E belongs to G_0, and each automorphism of G_0 is obtained by restriction of an inner automorphism of E.

In Chapter II the author extends partially the theory of maximal tori for connected compact groups to non-connected compact groups G. Let x be an element of G, N_x the identity component of the normalizer of x, T_0^h a maximal torus of N_x, and $T^{(h)}$ the closed subgroup of G generated by x and T_0^h. Then $T^{(h)}$ is the direct product of T_0^h and a finite cyclic group. If $x \in G_0$, the dimension $h = \operatorname{rank}(G_0)$. If x varies through some other component G_1 of G, h is constant but in general $h < \operatorname{rank}(G_0)$. However, the various subgroups $T^{(h)}$, $x \in G_1$, are all conjugate under G_0.

In Chapter III a diagram in the sense of Stiefel is introduced in each $T^{(h)}$ and the analog of the Weyl group is defined. A procedure is given for the construction of the diagram of $T^{(h)}$ from that of a maximal torus in G_0 and this is carried out in detail in the case when G_0 is simple. The result is applied to the extension $A(G_0)$ of $I(G_0)$, where G_0 is simple, to describe the involutive automorphisms of simple compact Lie algebras.

S. Helgason (Chicago, Ill.)

927:

İnönü, Erdal. Some remarks on the contraction of groups. Comm. Fac. Sci. Univ. Ankara. Sér. A. **8** (1956), 83–93. (Turkish summary)

This paper gives some examples of semi-direct products of two Lie groups. *S. Helgason* (Chicago, Ill.)

928:

Stoka, Marius I. Sur les groupes G_r mesurables du plan. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **9** (1957), 341–380. (Romanian. Russian and French summaries)

Let G_r be a Lie group operating in E_2. A necessary and sufficient condition that this transformation admit an invariant integral, unique up to a multiplicative constant, is transitivity and the vanishing of certain expressions in which the structure constants enter linearly as do certain two-rowed determinants formed from the components of the generating vector fields in E_2.

P. A. Smith (New York, N.Y.)

929:

Raševskiĭ, P. K. On linear representations of differential groups and Lie groups with nilpotent radical. Trudy Moskov. Mat. Obšč. **6** (1957), 337–370. (Russian)

The differential group D_v of class v (and order n) consists of all transformations $f: E_n \to E_n$ of cartesian n-space, where f is a polynomial function of degree $\leqq v$ without constant terms and with non-singular linear part, group operation being composition and subsequent reduction mod terms of degree $> v$. (In Ehresmann's terms, these are the v-jets at 0 of differomorphisms of E_n with 0 as fixed point.) D_v contains the unimodular linear group $\mathrm{SL}(n, \mathbf{R})$ as subgroup; its radical R consists of the maps with scalar linear part. Write D_v^0 for the subgroup with unimodular linear part; put $R_0 = R \cap D_v^0$; this is the radical of D_v^0, and is nilpotent. There is a natural representation of D_v on the space P_v of polynomials (over E_n) of degree $\leqq v$, by substitution and removal of terms of

degree $>v$. (One can interpret P_v as the space of C^∞-functions on E_n, modulo the functions whose derivatives of order $\leq v$ all vanish at 0.) The elements of this space are called covariant hypervectors [introduced by the author in 1928; cf. also H. V. Craig's extensors]. The conjugate space, with the contragredient action of D_v, is called the space of contravariant hyper-vectors; they can be interpreted as differential operators, at 0, of order $\leq v$. The author's first theorem states that all representations of D_v^0, up to equivalence, are given by invariant subspaces of contravariant hypertensor spaces (tensor products of several copies of the space of hypervectors). The proof begins by noting that for a representation φ of D_v^0 the induced representation of the subgroup $SL(n, \mathbf{R})$ splits into a number of representations on tensor spaces satisfying Young symmetry conditions. Each such tensor space is extended to the corresponding hypertensor space; this produces a new representation ψ of D_v^0. The main part of the proof consists in showing, with the help of a lemma on the possible linear relations between symmetric hypertensors, that ψ contains φ on an invariant subspace. The theorem can be considered as a classification of geometric objects of class $\leq v$. A version without the unimodular restriction is given. — The second theorem gives the structure of representations of a Lie group G with nilpotent radical R. Let S be a semi-simple Levi subgroup of G, and denote by G', R', S' the Lie algebras. Over R' one constructs the space T of systems of tensors of the form V, V_{α_1}, $V_{\alpha_1\alpha_2}$, $\cdots$, $V_{\alpha_1\cdots\alpha_m}$, satisfying the relations

$$V_{\alpha_1\cdots[\alpha_s\alpha_{s+1}]\cdots\alpha_k}=V_{\alpha_1\cdots\beta\cdots\alpha_k}c^\beta_{\alpha_s\alpha_{s+1}};$$

here the $c^\beta_{\alpha\gamma}$ are the structure constants of R'. A vector $a^\alpha \in R'$ is made to operate on T by $V_{\alpha_1\cdots\alpha_k}\to V_{\alpha_1\cdots\alpha_k}\cdot a^\alpha$. Further, S' operates on the ideal R', and hence, on T, in the natural fashion. It is shown that these two operations combine to a representation of G', and (locally) of G, on T. By tensoring T with representation spaces of S, one obtains certain representations of G. The theorem states that the invariant subspaces of these representations give all complex-indecomposable unimodular representations of G. By operating on the coordinates of a vector with the basis elements of R', one gets quantities of the type $V_{\alpha_1\cdots\alpha_k}{}^\iota$. One verifies that in terms of these quantities the representation takes the promised form. This generalizes a result of Berezin [Dokl. Akad. Nauk SSSR (N.S.) **93** (1953), 759–761; MR **15**, 600].

H. Samelson (Ann Arbor, Mich.)

930:

Dieudonné, Jean. Groupes de Lie et hyperalgèbres de Lie sur un corps de caractéristique $p>0$. V. Bull. Soc. Math. France **84** (1956), 207–239.

L'auteur définit en caractéristique $p\neq 0$ un analogue à la série exponentielle. Soient K un corps de caractéristique $p\neq 0$ et $\mathbf{U}=(U_n)_{n\geq 0}$ une suite d'indéterminées non commutatives, où U_n a le poids p^n; il existe alors des polynomes Z_n de poids n en les U_i ayant les propriétés suivantes: a) $Z_{p^s}=U_n$; b) si l'on pose $E(\mathbf{U})=\sum_{n\geq 0}Z_n(\mathbf{U})$ et $W_n=U_n+V_n+\sum_{0<i<p^s}Z_i(\mathbf{U})\cdot Z_{p^s-i}(\mathbf{V})$, on a $E(\mathbf{U})\cdot E(\mathbf{V})=E(\mathbf{W})$, avec $\mathbf{V}=(V_n)_{n\geq 0}$ et $\mathbf{W}=(W_n)_{n\geq 0}$, pourvu que l'on ait $U_iV_j=V_jU_i$ pour $i, j\geq 0$. Si l'on omet cette dernière restriction, il existe des séries $Y_n(\mathbf{U}, \mathbf{V})$ telles que $E(\mathbf{U})\cdot E(\mathbf{V})=E(\mathbf{Y})$ avec $\mathbf{Y}=(Y_n)_{n\geq 0}$ et où tous les termes de Y_n sont de poids $\geq p^n$ (analogue de la formule classique de Hausdorff); mais l'auteur ne démontre pas l'unicité des séries Y_n une fois $E(\mathbf{U})$ fixée. Lorsque les U_n commutent deux à deux, $E(\mathbf{U})$ se réduit essentiellement à l'hyperexponentielle introduite par l'auteur dans Rend. Circ. Mat. Palermo (2) **1** (1952), 380–402 [MR **14**, 1062]. Les démonstrations utilisent la théorie des hyperalgèbres; soient A une algèbre et Δ un homomorphisme d'algèbre de A dans $A\otimes A$; sur le dual A^* de A, on définit une multiplication par $f\cdot g=(f\otimes g)\circ\Delta$, et si A^* pour cette multiplication est isomorphe à une algèbre de séries formelles à un nombre fini ou infini de variables, on dit que (A, Δ) est une hyperalgèbre. Si (A, Δ) est une hyperalgèbre, il existe (th. 2) des suites $\mathbf{U}^{(i)}=(U_h^{(i)})_{h\geq 0}$ d'éléments de A telles que les produits finis $\prod_i Z_{\alpha_i}(\mathbf{U}^{(i)})$, où l'ensemble des i est totalement ordonné, forment une base de A sur K. De plus, l'auteur définit des hyperalgèbres $\mathfrak{F}_1(K)$ (algèbre des polynomes non commutatifs en les U_n, avec $\Delta(U_n)=\sum_{0\leq i\leq n}Z_i(\mathbf{U}\otimes 1)\cdot Z_{n-i}(1\otimes\mathbf{U})$) au th. 3, et $\mathfrak{H}_1(K)$ (algèbre des polynomes non commutatifs en les V_n, avec $\Delta(V_n)=\sum_{0\leq i\leq n}V_i\otimes V_{n-i}$) au th. 1; il définit même des hyperalgèbres $\mathfrak{F}_n(K)$ et $\mathfrak{H}_n(K)$ qui sont des produits libres de n copies des précédentes.

Parmi les erreurs d'impression, relevons les suivantes: $<$ et non $\leq$, p. 208, l. 9; "linéaire" et non "semi-linéaire pour $\xi\to\xi^{p-1}$" à la p. 210, l. 8 et 9; ajouter "et $\alpha\neq 0$" après "et $f\in\mathfrak{o}_r$" à la l. 3 du bas de la p. 210.

P. Cartier (Princeton, N.J.)

931:

Dieudonné, Jean. Lie groups and Lie hyperalgebras over a field of characteristic $p>0$. VI. Amer. J. Math. **79** (1957), 331–388.

To every algebraic group G it is possible to attach a formal Lie group G^*: if $z_1, \cdots, z_n$ form a system of uniformizing parameters at the neutral element of G, the functions $(s, t)\to z_i(st)$ may be expanded in formal power series in the $z_i(s)$, $z_j(t)$, and these formal power series define a formal Lie group G^*; if the z_i's are replaced by another system of uniformizing variables, G^* is replaced by an isomorphic group. Let G now be any formal Lie group; the aim of the present paper is to derive as many properties of G as possible from the known properties of algebraic groups and from the study of homomorphisms of G into formal Lie groups associated to algebraic groups. However, results about algebraic groups consist mostly in assertions about the existence, structure and conjugacy properties of certain subgroups (maximal solvable subgroups, maximal tori, Cartan subgroups; cf. A. Borel, Ann. of Math. (2) **64** (1956), 20–82 [MR **19**, 1195]). In order to formulate the analogous statements for formal groups, the author first has to define the notions of subgroup, normal subgroup, kernel, etc. for formal groups; the elementary functorial properties of these objects are here by no means trivial because we lack the set-theoretic substratum of the ordinary group theory. A subgroup of a formal Lie group G is defined to be a pair (H, f), where H is a formal Lie group and f a monomorphism of H into G, with the obvious identification between such pairs. The subgroups correspond in a one-to-one manner to certain (but not all) subalgebras of the hyperalgebra of G; the author gives a direct characterization of those subalgebras (called subhyperalgebras) which correspond to subgroups. The definition of normal subgroups is patterned on the classical one; on the other hand, any homomorphism of G defines a subgroup of G, its kernel; it is proved that normal subgroups and kernels of homomorphisms are the same objects. The homomorphism theorems of Fr. Noether are generalized to the case of formal Lie groups; there is a modified Jordan-Hölder theorem in which isomorphisms between quotients have to be replaced by

isogenies. The center, derived groups, etc. can be defined in the usual manner for formal groups.

A formal Lie group G is called representable if there exist a linear algebraic group H and an isogeny of G on a formal subgroup of the formal group associated to H. The author proves, by a construction which generalizes the adjoint representation, that, if Z is the center of an arbitrary formal Lie group G, then G/Z is representable. If u is a homomorphism of a formal Lie group G into the formal group H^* associated to an algebraic group H, there is a largest algebraic subgroup H' of H which "contains" $u(G)$; H' is called the algebraic hull of G, and is denoted by $\mathfrak{A}(G, u)$ or $\mathfrak{A}(G)$. If u is a monomorphism, the mapping $M \to \mathfrak{A}(M)$ (for subgroups M of G) is compatible with the most important properties of subgroups (for instance of being normal, or commutative, or solvable, etc.). Using homomorphisms into algebraic groups, the author establishes the existence of the normalizer for any subgroup of a formal group G, and he proves that the subgroups of G form a complete lattice with respect to inclusion.

At this stage, it becomes at last possible to translate for formal groups the theorems of the Borel theory of algebraic groups. For lack of a notion of conjugacy in an arbitrary formal group, the conjugacy theorems are stated for subgroups of the formal group $GL^*(n)$ associated to $GL(n)$: conjugacy then means conjugacy by an element of the algebraic hull of the group. On the other hand, conjugacy theorems have also a weaker form valid in the formal group itself; thus, all maximal solvable subgroups, or tori, of a formal group are isomorphic to each other. This being said, all the main theorems of Borel have their counterpart in formal Lie groups; it would be too long to describe here these substitutes for each theorem separately.

A formal Lie group G is called semi-simple if it has no commutative normal subgroup of dimension >0. In that case, not only is G representable, but there is an isogeny of G on the whole formal group associated to a semi-simple linear algebraic group; thus, classification problems for semi-simple formal Lie groups are reduced to similar problems for algebraic groups. *C. Chevalley* (Paris)

932:

Dieudonné, Jean. Les algèbres de Lie simples associées aux groupes simples algébriques sur un corps de caractéristique $p>0$. Rend. Circ. Mat. Palermo (2) **6** (1957), 198–204.

Let g be a simple finite dimensional Lie algebra over the field of the complex numbers. A normalized basis for g, in the sense of Chevalley [Tôhoku Math. J. (2) **7** (1955), 14–66; MR **17**, 457], for which the multiplication table is integral, yields a Lie algebra g^* over the ring Z of the rational integers (having for a basis a normalized basis of g), and g^* is determined up to an isomorphism by g. Hence, if K is any field, of arbitrary characteristic, one can invariantly associate with g a Lie algebra $g_K = g^* \otimes K$ over K. The question arises of how far these Lie algebras g_K are from being simple. For the classes A, B, C, D of the classical simple Lie algebras g, the determination of the ideals and composition series of the g_K's can easily be extracted from of results of Jacobson [Amer. J. Math. **63** (1941), 481–515; Trans. Amer. Math. Soc. **50** (1941), 15–25; MR **3**, 103]. Here, this determination is accomplished for the five exceptional simple Lie algebras g by direct computation.

The result is that g_K, with g exceptional and K arbitrary, is simple, except in the following cases: 1) g is of type G_2 and K is of characteristic 3; then g_K has only a single proper ideal u, which is of dimension 7, and the Lie algebras u and g_K/u are simple and isomorphic. 2) g is of type F_4 and K is of characteristic 2; then g_K has only a single proper ideal u, which is of dimension 26, and the Lie algebras u and g_K/u are simple and isomorphic. 3) g is of type E_6 and K is of characteristic 3; then the only proper ideal of g_K is its 1-dimensional center c, and g_K/c is simple. 4) g is of type E_7 and K is of characteristic 2; then the only proper ideal of g_K is its 1-dimensional center c, and g_K/c is simple. *G. Hochschild* (Berkeley, Calif.)

933:

Curtis, Charles W. Modular Lie algebras. II. Trans. Amer. Math. Soc. **86** (1957), 91–108.

The author here completes his program, begun in Part I [same Trans. **82** (1956), 160-179; MR **18**, 51], for constructing all Lie algebras with non-degenerate Killing forms over algebraically closed fields of characteristic $p>7$. Using the work of Chevalley [Tôhoku Math. J. (2) **7** (1955), 14–66; MR **17**, 457], the procedures of his earlier paper are simplified. On the basis of results of Jacobson and the reviewer [Seligman, Mem. Amer. Math. Soc. no. **19** (1956), §§ 1–5; MR **17**, 1108], each such algebra can be assigned a "Weyl matrix" (a_{ij}) of rational integers (here obtained by showing that the roots can be lexicographically ordered in such a way that simple systems of roots readily appear). Any Weyl matrix obtained for the algebra L satisfies the conditions of Harish-Chandra [Trans. Amer. Math. Soc. **70** (1951), 28–96; MR **13**, 428], therefore is the Weyl matrix of a unique complex semi-simple Lie algebra. The procedures of Part I then yield a uniquely determined Lie algebra over the original algebraically closed modular field, and this algebra is proved isomorphic to L. The methods are unified in that the classification of Weyl matrices is not used; the reduction of the classification problem for the algebras to the classification of Weyl matrices is a corollary of the main result. *G. B. Seligman* (Muenster)

934:

Mycielski, Jan. On the extension of equalities in connected topological groups. Fund. Math. **44** (1957), 300–302.

Let G be a topological group whose identity element is denoted 1. A word on G is a function φ from $G \times \cdots \times G$ to G of the type $\varphi(x_1, \cdots, x_n) = x_{s_1}^{k_1} \cdots x_{s_m}^{k_m}$ where no formal cancellation is possible. If in addition fixed elements of G are permitted, then φ is called a word with constants on G. Consider the following properties which the group G might have: (A) If φ and ψ are words in n variables with constants and the relation $\varphi(x_1, \cdots, x_n) = \psi(x_1, \cdots, x_n)$ holds for $\langle x_1, \cdots, x_n \rangle$ in an open subset of G^n, then the same relation holds throughout G^n. (A') If φ is a word in one variable with constants and $\varphi(x) = 1$ for all x in an open subset of G, then the same is true for all x in G. It is proved that (A) and (A') are equivalent properties and that they are enjoyed by Lie groups, locally compact connected groups, and connected abelian groups. The question for arbitrary connected groups is open. *A. M. Gleason* (Cambridge, Mass.)

935:

Balcerzyk, S.; and Mycielski, Jan. On the existence of free subgroups in topological groups. Fund. Math. **44** (1957), 303–308.

In the notation of the preceding review, a group is said to be 'functionally free' if for each non-trivial word φ in

two variables, there exist $\xi, \eta \in G$ such that $\varphi(\xi, \eta) \neq 1$. Theorem: Every topological group with property (A), functionally free and such that G and $G \times G$ are not of the first category on themselves, contains a free subgroup of rank $\aleph_1$. Theorem: Every locally compact, connected, non-solvable group G contains a free subgroup of rank $2^{\aleph_0}$. Theorem: Every locally compact non-0-dimensional group G contains a free abelian subgroup of rank $2^{\aleph_0}$. These results imply that if, in a connected locally compact group G, there is a non-trivial word Q in two variables which takes only the value 1, then G is solvable. The author proposes the interesting question: Can one find the length of the commutator chain of G from φ?

A. M. Gleason (Cambridge, Mass.)

TOPOLOGICAL ALGEBRA

See 823, 824, 825.

FUNCTIONS OF REAL VARIABLES

See also 1052, 1061, 1123, 1165.

936:

***Osgood, William Fogg. Functions of real variables; Functions of a complex variable. Bound in one volume.** Chelsea Publishing Co., New York, N.Y., 1958. xii+407 pp.; viii+262 pp. $4.95.

A reproduction of the two well-known works first published in Peiping in 1936 [Univ. Press].

937:

Timan, A. F. Addendum to the article of A. B. Efimov "Estimation of the modulus of continuity of functions of class H_2^1". Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 595–598. (Russian)

The result of the paper of Efimov [Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 283–288; MR **19**, 534] is improved by extending it to arbitrary continuous functions on $(-\infty, +\infty)$ instead of the periodic ones.

G. G. Lorentz (Syracuse, N.Y.)

938:

Froda, Alexandru. Propriétés (à distance) des fonctions réelles dans un espace euclidien. Acad. R. P. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **8** (1956), 683–686. (Romanian. Russian and French summaries)

Romanian version of C. R. Acad. Sci. Paris **242** (1956), 1948–1951 [MR **17**, 953].

M. E. Shanks (W. Lafayette, Ind.)

939:

Dobrescu, Eugen V. Contribution à une analyse infinitésimale bidimensionnelle. Acad. R. P. Romîne. Stud. Cerc. Mat. **8** (1957), 103–130. (Romanian. Russian and French summaries)

The author starts from an idea attributed to K. Bögel [J. Reine Angew. Math. **170** (1934), 197–217; **173** (1935), 5–30]: instead of using a neighborhood topology in two dimensions, use the corners of rectangles as the generalization of the endpoints of intervals in one dimension. Thus $f(x, y)$ is "globally continuous" at (x_0, y_0) if its second differences starting from (x_0, y_0) approach zero. The author defines global boundedness and global semi-continuity analogously and proves a number of theorems involving these concepts. Then he turns to double series: $\sum\sum a_{mn}$ is globally convergent if for any positive ε, p, q, $|a_{m+p,n+q} - a_{m+p,n} - a_{m,n+q} + a_{mn}| < \varepsilon$ provided $m \geq M(\varepsilon)$, $n \geq N(\varepsilon)$. He gives convergence criteria and examples. [Cf. M. Nicolesco, Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **42** (1940), 53–56; MR **7**, 13; and the comments made in the review.] Finally the author considers globally convergent series of functions and the relation between global continuity or global semi-continuity of the terms and of the sum.

R. P. Boas, Jr. (Evanston, Ill.)

940:

***Hilton, P. J. Differential calculus.** Library of Mathematics. The Free Press, Glencoe, Ill., 1958. vii+56 pp. $1.25.

"This book is intended to provide the university student in the physical sciences with information about the differential calculus which he is likely to need... The emphasis is on detailed discussion of the ideas... rather than on complete proofs." *From the preface*

941:

Nilov, G. N. Expansion of a continuous function in a generalized power series. Kabardin. Gos. Ped. Inst. Uč. Zap. **8** (1955), 25–27. (Russian)

942:

Dubovickiĭ, A. Ya. On the structure of level sets of differentiable mappings of an n-dimensional cube into a k-dimensional cube. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 371–408. (Russian)

Let $u_i = u_i(x_1, \cdots, x_n)$ $(i = 1, \cdots, k)$ define a mapping U of an n-dimensional cube C_n into a k-dimensional cube C_k. Let the functions u_i be differentiable through order $n-k+1-s$. Let M be the critical set of the mapping, i.e., the set on which the Jacobian matrix has less than maximum rank. The author proves: except for a set of k-dimensional measure zero, the inverse image of each point u in C_k meets M in a set whose s-dimensional measure is zero. (The measures are Hausdorff measures). From this, it follows that, if $s=0$, then $U(M)$ has k-dimensional measure zero. These results supplement those of an earlier paper by the author [Mat. Sb. N.S. **32(74)** (1953), 443–464; MR **15**, 299] and are closely related to theorems of Sard [Bull. Amer. Math. Soc. **48** (1942), 883–890; MR **4**, 153]. *W. Kaplan* (Ann Arbor, Mich.)

943:

Konstantinesku, F. A new proof of a theorem of V. A. Markov using a theorem of Sturm. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 6(78), 147–148. (Russian)

D'après W. A. Markoff [Math. Ann. **77** (1916), 213–258; p. 252], l'hypothèse que les zéros des polynomes $P(x)$ et $Q(x)$ sont réels et séparés entre eux entraîne la même conclusion pour les dérivées $P'(x)$ et $Q'(x)$. L'auteur donne une nouvelle démonstration de ce fait en remarquant que ceci résulte aussi du théorème d'oscillation de Sturm relatif à l'équation diff. linéaire du second ordre.

M. Tomić (Beograd)

944:

Downing, H. H.; and Jasper, S. J. On homogeneous functions. Bul. Inst. Politech. Iaşi **4** (1949), 58–62.

Five theorems which are generalizations of Euler's theorem or its converse.

945:

Turán, Paul. Remark on the theory of quasianalytic function-classes. Magyar Tud. Akad. Mat. Kutató Int. Közl. 1 (1956), 481–487 (1957). (Hungarian and Russian summaries)

L'auteur considère la quasi-analyticité dans le sens du reviewer. Une classe de fonctions localement intégrables L est quasi-analytique (α) dans ce sens, si deux fonctions $f_1(x)$ et $f_2(x)$ de cette classe satisfaisant à la relation $\liminf_{h\to+0} \exp(h^{-\alpha}) \int_{x_0-h}^{x_0} |f_1(x)-f_2(x)|dx<\infty$ sont nécessairement égales p.p. Il y a quelques années l'auteur [voir par ex. "Eine neue Methode in der Analysis und deren Anwendungen", Akadémiai Kiadó, Budapest, 1953; MR 15, 688] a pu, en utilisant ses méthodes d'évaluation du maximum d'un polynome trigonométrique par son maximum sur un sous-intervalle, introduire un nouveau critère d'une telle quasi-analyticité. Tandis que le reviewer caractérise une telle quasi-analyticité par la "lacunarité" de la série de Fourier [Séries de Fourier et classes quasi-analytiques, Gauthier-Villars, Paris, 1935], procédé généralisé par B. Ya. Levin pour les fonctions presque-périodiques [Doklady Akad. Nauk SSSR (N.S.) 65 (1949), 605–608; MR 11, 23], l'auteur caractérise la classe par

$$(1) \qquad f(x)=\sum_1^\infty a_j \exp(it_jx),$$

$$(2) \qquad \limsup_{\omega\to\infty} \exp(2\alpha^{-1}\omega \log \omega) \sum_{t_j>\omega} |a_j|<\infty.$$

L'auteur a démontré que si

$$(3) \qquad \liminf_{h\to+0} \exp(h^{-\alpha}) \max_{x_0-h\le x\le x_0} |f_1(x)-f_2(x)|<\infty,$$

f_1 et f_2 appartenant à la classe, alors $f_1=f_2$ p.p. L'auteur cherche maintenant à démontrer que son résultat ne peut pas être beaucoup amélioré. Ainsi, la condition (2) ne peut pas être remplacée par la condition

$$(4) \qquad \limsup_{\omega\to\infty} \exp(\omega^{1-\epsilon}) \sum_{t_j>\omega} |a_j|<\infty.$$

En effet, en posant $\beta=2[1/\varepsilon]>\max(2,\alpha)$, β pair, la fonction $f(x)=\exp(-1/\sin^\beta x)$, tout en satisfaisant aux conditions (1), (4) et la relation $\liminf_{h\to+0} \exp(h^{-\alpha}) \max|f(x)|<\infty$, n'est pas identiquement nulle.

S. Mandelbrojt (Paris)

MEASURE AND INTEGRATION

See also 942, 1196, 1215, 1219, 1347.

946:

Phakadze, Š. S. On extension of measure. Soobšč. Akad. Nauk Gruzin. SSR 17 (1956), 769–776. (Russian)

A class M of subsets of the Euclidean space R^n is called an invariant class if M is a congruence-invariant σ-algebra containing the unit cube R_0^n. An invariant measure μ on M is a σ-additive invariant measure satisfying $\mu(R_0^n)=1$. μ is extendible if there is another invariant measure μ' on an invariant class $M'\supset M$, $M'\neq M$, such that $\mu'=\mu$ on M. Sierpinski posed the problem whether any invariant measure is extendible. Th. 1 of the paper gives an affirmative solution of this problem, under the hypothesis that any inaccessible cardinal number is $>$ than the continuum. Th. 2: If μ is an invariant measure on M and $\mu(E)>0$, then there exists a non-μ-measurable set $E'\subset E$.

μ is said to be of type (B) if for every set $A\in M$, $\mu(A)>0$, there exists a sequence of sets $A_i\in M$ such that $\mu(A_0)=0$, $A_i\simeq A_i'\subset A$ ($i\ge 1$) and $\sum_0^\infty A_i=R^n$. Th. 4: any invariant measure μ on M of type (B) is extendible. Th. 5: If μ is an uniquely determined invariant measure on M then μ is of type (B). Th. 3: If μ is a complete and invariant measure of type (B) on M, then any other invariant measure μ' on M coincides with μ.

M. Cotlar (St. Louis, Mo.)

947:

Ishi, Todashi. The (MA)-condition for linear functionals. Sûgaku 8 (1956/57), 153–157. (Japanese)

Let X be a completely regular space, and C a vector lattice of real-valued continuous functions which satisfies: $f\in C$ implies $1\wedge f\in C$, and for every closed set F and every point $p\notin F$, there exists $f\in C$ such that $0\le f\le 1$, $f(p)=1$ and $f=0$ on F.

The author shows that every positive linear functional on C which satisfies the (MA)-condition, that is, which is continuous by order-topology, has an integral representation by a Borel measure which is reducible in the sense of Katětov [Fund. Math. 38 (1951), 73–84; MR 14, 27].

This generalizes the known result in case X is locally compact and C the totality of all continuous functions which vanish outside of some compact set.

I. Amemiya (Kingston, Ont.)

948:

Ishi, Todashi. The (MA)-condition for linear functionals. II. Sûgaku 8 (1956/57), 213–215. (Japanese)

Let C be the space of all real-valued continuous functions (on a completely regular space X). Hewitt [Fund. Math. 37 (1950), 161–189; MR 13, 147] showed that every positive linear functional I on C has an integral representation by a Baire measure γ.

Here the author gives a necessary and sufficient condition for γ in order that I satisfy the (MA)-condition, and some relations between γ and the Borel measure obtained in the paper reviewed above. He also proves that X is a Q-space [compact space] if and only if every positive linear functional on C [on C^*=the totality of bounded functions of C] satisfies the (MA)-condition in C [C^*].

I. Amemiya (Kingston, Ont.)

949:

Natanson, I. P.; and Natanson, G. I. On the mutual relation between the integral of Denjoy in the narrow and in the wide sense. Uspehi Mat. Nauk (N.S.) 12 (1957), no. 6(78), 161–168. (Russian)

Let integral operators A and B be defined for real functions on a real interval $[a, b]$. Then $A\subset B$ if every A-integrable function is B-integrable and the integrals are equal. (The notations A and B will also designate the domains of the operators.) For an integral operator T, extensions T^* and T_* are defined. $T_*\subset T^*$. Let $D_0=L$, the domain of the Lebesgue integral. For every $\xi<\Omega$, let $D_{\xi+1}=(D_\xi)_*$ and, if ξ is a limiting ordinal, let $D_\xi=\sum_{\eta<\xi} D_\eta$; define D^ξ in similar fashion (using T^* instead of T_*). The integrals are then extended to these domains in standard fashion. Now $D_*=\sum_{\xi<\Omega} D_\xi$ and $D^*=\sum_{\xi<\Omega} D^\xi$ are the domains of the Denjoy integral in the narrow sense and the wide sense, respectively. The following are proved.

a) For every $\xi<\Omega$, $D_\xi\subset D^\xi$.

b) For every $\xi<\Omega$, there are functions belonging to $D_{\xi+1}$ which do not belong to D^ξ.

c) For every $\xi<\Omega$, there are functions in $D^{\xi+1}$ which are neither in D^ξ nor in $D_{\xi+1}$.

C. Goffman (Lafayette, Ind.)

950:

Ridder, J. Integration von Differentialkoeffizienten höherer Ordnung. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 364–368.

Given, concerning a continuous real-valued function f on a real interval $[a, b]$, its nth differential coefficient f_n in a certain generalized sense, the author indicates how, by n successive Cesàro-Perron integrations of successive integer orders, the function f may be recovered, up to an arbitrary polynomial of degree $n-1$, from f_n.

T. A. Botts (Charlottesville, Va.)

951:

Urbanik, K. Remarks on the Doss integral. Colloq. Math. **5** (1957), 95–102.

Es sei $\mathfrak{X}$ ein metrischer Raum mit der Metrik ρ und m ein Wahrscheinlichkeitsmaß in einer Menge I. Eine meßbare Abbildung f von I in $\mathfrak{X}$ heißt Doss-integrierbar, wenn es ein und nur ein Element a in $\mathfrak{X}$ gibt, so daß für jedes z aus $\mathfrak{X}$ gilt $\rho(a, z) \leq \int_I \rho(f(t), z)dm(t)$; man setzt dann $a=\int_I f dm$. Stellt $\mathfrak{X}$ insbesondere einen normierten linearen Raum dar und nimmt f nur endlich viele Werte $x_1, \cdots, x_n$ auf den Mengen $I_1, \cdots, I_n$ an, so ist f offenbar Doss-integrierbar mit (*) $\int_I f dm=\sum_j x_j m(I_j)$. Der Verf. zeigt nun umgekehrt: Bildet $\mathfrak{X}$ eine kommutative metrische Gruppe mit invarianter Metrik und ist jede nur endlich viele Werte annehmende meßbare Funktion Doss-integrierbar, so ist $\mathfrak{X}$ sogar ein normierter linearer Raum. Ist ρ, bei sonst unveränderten Voraussetzungen, nicht invariant, so braucht es, wie ein Beispiel zeigt, keine skalare Multiplikation zu geben, mit der die Formel (*) stets gilt.

K. Krickeberg (Würzburg)

952:

Leĭfman, L. Ya. On conditions for existence of a Kolmogoroff integral and the concept of differential equivalence. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 343–352. (Russian)

The integrals are those defined by Kolmogoroff [Math. Ann. **103** (1930), 654–696]. If f is a (possibly multivalued) set function defined for all sets of a family $\mathfrak{M}$, which are subsets of a set E_0, the multiplicative integral $\int f(dE)$ is the limit of the sums $\sum f(E_n)$, where the sums are taken over partitions of E into a countable number of sets in $\mathfrak{M}$ and where the limit is the directed limit as the partitions are refined. A starred integral, in which only finite partitions are allowed, is discussed. Two functions f_1 and f_2 are differentiably equivalent for $\mathfrak{M}$ if $\int |f_1(dE)-f_2(dE)|=0$. The author gives conditions for existence of the integral and for differential equivalence: these are in terms of the corresponding upper and lower integrals for the functions $\bar{f}, \underline{f}, \bar{f}(E)=\sup f(E)$, and are of the forms that would be anticipated from the theory of the Riemann integral.

J. L. B. Cooper (Cardiff)

953:

Radó, T. Lebesgue area and Hausdorff measure. Fund. Math. **44** (1957), 198–237.

One task of area theory is to express the Lebesgue area, of a map T from a square Q into 3-space, as the integral of a suitable multiplicity $k(x, T)$ with respect to Hausdorff 2-dimensional measure H. This entails diminishing drastically the crude multiplicity obtained by counting the points of the set of models of x under T. The first step is normally to count, not the points, but the maximal model continua (mmc) under T interior to Q, and it then remains somehow to trim down their aggregate to the barest essentials. For a plane map, a system of essential mmc was devised in the author's "Length and area" [Amer. Math. Soc. Colloq. Publ., vol. 30, New York, 1948; MR 9, 505].

For any open set O in Q, let $O(P)$ denote the intersection of O with the union of the essential mmc under the plane map derived from T by orthogonal projection into a plane P and let $O^*(P)$ be the union of those essential mmc under it which are interior to O. We write further $H_P E=\inf H(E-S)$, where the inf is for all H-measurable sets S whose projections on P have plane measure 0. An mmc under T will be termed significant for x, if, for each neighbourhood O of this mmc, the sets $X_P=TO(P)$ satisfy the lower density condition $\liminf D_r>0$ as $r\to 0$, where D_r is the sup in P of $r^{-2}H_P(X_P \cap K_r)$ and K_r denotes the ball of centre x and radius r. E. J. Mickle [Trans. Amer. Math. Soc. **82** (1956), 440–451; MR **18**, 24] defined a multiplicity $k(x, T)$ as the number of mmc under T significant for x. It possesses several desirable properties and its integral with respect to H is the desired Lebesgue area. Unfortunately it may lack a "local" character, such as would have been attained if the sets X_P in the lower density condition had been of the form $TO^*(P)$; this would have resulted in a still further trimming, which might well spoil everything: in fact, every satisfactory opportunity of trimming may have had to be used up in Mickle's highly ingenious construction, and there is little room for a modification with an added property of local character.

Nevertheless the author achieves such a modification and his end result differs perhaps only slightly from that of Mickle. To reach it, he has to throw everything back into the melting pot, even the first step: significant sets are no longer mmc, nor even disjoint, and $k(x, T)$ is derived, not by counting, but by a peculiar supremum rule. In fact the author has stripped all but the barest essentials from the argument of Mickle, rather than from a set of mmc, and the resulting gain in flexibility makes it possible to obtain the desired, more satisfactory, $k(x, T)$. The whole approach represents a new departure in the subject, and it leads to a number of intriguing further problems.

L. C. Young (Cambridge, England)

954:

Cesari, L.; and Turner, L. H. On the concept of surface integral. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 42–43.

By a surface $S=(T, A)$ is meant a continuous mapping T from an admissible set A of E_2 into E_3. Cesari [Ann. Scuola Norm. Sup. Pisa (2) **13** (1944), 77–117; MR **9**, 505] has defined the surface integral

$$J(T, A, f)=\int_S f(\phi, t),$$

for any surface of finite area and for any f on $T(A)\times S^2$ to the reals, which is uniformly continuous, S^2 being the unit sphere $t_1^2+t_2^2+t_3^2=1$ in E_3. Here $\phi=T(w)$ for $w \in A$. This definition has since been fundamental for problems of the calculus of variations for parametric surfaces. It generalized, for example, the Tonelli integral

$$I(T, A, f)=\int_A f[T(w), J(w)]dw,$$

where $J(w)$ is the generalized Jacobian at w, to the general situation in which the latter does not exist.

In the present paper the authors construct, from the Cesari variation functions, a measure ϕ and new set functions V_1, V_2, V_3. The Radon-Nikodym derivatives

$\theta_\nu = dV_\nu/d\phi$ exist always, and $\theta_1^2+\theta_2^2+\theta_3^2=1$ almost everywhere ϕ. The integral

$$H(T, A, f) = \int_A f[T(w), \theta(w)]d\phi,$$

where $\theta(w)=(\theta_1(w), \theta_2(w), \theta_3(w))$, then exists for every surface of finite area and every f bounded and B-measurable in $T(A)\times S^2$. The principal theorem is that then

$$J(T, A, f) = H(T, A, f).$$

H thus gives a definition of surface area and an integral representation of J valid for every surface of finite area.

J. M. Danskin (New Brunswick, N.J.)

955:

Michael, J. H. Integration over parametric surfaces. Proc. London Math. Soc. (3) **7** (1957), 616–640.

A surface is defined as a pair (f, M_n) consisting of a compact n-dimensional manifold M_n and a continuous f on M_n into euclidean k space R_k, $k\geq n$. For $k=n$, $A\subset M_n$, $x\in R_n$, $-f(\mathrm{fr}(A))$, the degree $d(f, A, x)$ is defined, in more or less standard fashion. For the case $k=n+1$ and M_n closed, the order $u(x)=u(f, M_n, x)$ is defined for all $x\in R_{n+1}-f(M_n)$ as $d(P_1f, A, P_1(x))$; where $A\subset M_n$ is such that $\overline{f(A)}$ and $\overline{f(M_n-A)}$ do not meet $L_{1-}(x)$ and $L_{1+}(x)$, respectively, $P_1(x_1, \cdots, x_{n+1})=(x_2, \cdots, x_{n+1})$, $L_{1-}(x)=\{x+\lambda d_1, \lambda\leq 0\}$, $L_{1+}(x)=\{x+\lambda d_1, \lambda\geq 0\}$, and $d_1=(1, 0, \cdots, 0)$. The notation $O=O(f, M_n)$ is used to designate the set for which $u(x)\neq 0$.

For flat surfaces (mappings of M_n into R_n) a measure is defined in M_n as follows: A shell is the union of a finite number of sets in R_n of the form $\{(x_1, \cdots, x_n);\ a_i\leq x_i\leq b_i\}$, with at least one $a_i=b_i$. $\mathfrak{R}$ is the ring of sets $A\subset M_n$ such that $f(\mathrm{fr}(A))$ is contained in a shell. For every $A\in\mathfrak{R}$, the number $\mu(A)=(R_n-f(\mathrm{fr}(A))\int d(f, A, x)dx$ is defined, as well as the associated variations μ_+, μ_-, and $\bar{\mu}$. For every $A\in\mathfrak{R}$, $x\in R_n-f(\mathrm{fr}(A))$, define $e(f, A, x)=\sup\sum_{i=1}^{p} d(f, C_i, x)$, where the C_i are subsets of A with disjoint interiors, and $x\notin f(\mathrm{fr}(C_i))$. It is shown that for every $A\in\mathfrak{R}$, $\bar{\mu}(A)=(R_n-f(\mathrm{fr}(A))\int e(f, A, x)dx$. f is said to be of bounded variation if μ is bounded on $\mathfrak{R}$. It is shown that if f is of bounded variation, $B\subset M_n$, and $f(\mathrm{fr}(B))$ has Lebesgue measure 0, then B is μ_+^*, μ_-^* measurable (where * designates induced outer measure) and $\mu_+(B)-\mu_-(B)=(R_n-f(\mathrm{fr}(B))\int d(f, B, x)dx$. For a closed n-surface (f, M_n) in R_k, $k>n$, and a projection P onto a coordinate n-plane, the surface integral $(f, A)\int g(x)dP(x)$ is defined as the Lebesgue-Stieltjes integral $(A)\int gfd\mu(Pf)=(A)\int gfd\mu_+(Pf)-(A)\int gfd\mu_-(Pf)$, provided Pf has bounded variation on M_n and the function gf is integrable with respect to $\mu_+(Pf)$ and $\mu_-(Pf)$.

The following general form of the Gauss-Green theorem is proved: If g, real-valued, is defined and continuous on an open set containing $f(M_n)\cup O$, if $\partial g/\partial x_j$ exists and $u(x)\partial g(x)/\partial x_j$ is integrable on O, with respect to x, then $(O)\int u(x)(\partial g/\partial x_j)dx=(-1)^{j-1}(f, M_n)\int g(x)dP_j(x)$.

{The author mentions that the relation between the surface integral of this paper and one given by Cesari for mappings of certain subsets of R_2 into R_3 has not been studied. It thus seems appropriate that the reviewer comment on this relation. The integrands discussed by Cesari are not merely functions of position but also of "direction", a fact which leads to formidable difficulties, indeed. The present author deals with point functions and, essentially, with flat mappings. For the case of mappings of R_2 into R_3 his theory of integration becomes a standard topic of surface area theory. The merit of his theory seems to be in the arbitrariness of the dimension and generality of the domain.}

C. Goffman (Lafayette, Ind.)

FUNCTIONS OF A COMPLEX VARIABLE

See also 989, 1163, 1169, 1349, 1434.

956:

***Franklin, Philip. Functions of complex variables.** Prentice-Hall Mathematics Series. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1958. ix+246 pp. $6.95.

The book under review is intended as an elementary text for a first course in complex variables at either the senior or first year graduate level. For students with an adequate background in advanced calculus the present text could be completed in one semester. In the reviewer's opinion this book achieves its objective of being an adequate introduction to the subject both for those whose interest is in the applications as well as those whose interest is primarily mathematical.

The first six chapters are devoted to geometric function theory. The author gives a detailed treatment to the bilinear transformation together with an introduction to the geometry of inversion. The latter topic, though not usually found in an elementary text, gives the student a clearer view of the significant geometric properties of the bilinear transformation. In the remaining four chapters the author develops the usual consequences of the Cauchy Integral Theorem. The large number of drill exercises and exercises intended to extend the text will thoroughly test the student's mastery of the material. Some typographical errors were noted but the reviewer found none that the student could not easily correct.

V. F. Cowling (Lexington, Ky.)

957:

Peschl, Ernst. Les invariants différentiels non holomorphes et leur rôle dans la théorie des fonctions. Rend. Sem. Mat. Messina **1** (1955), 100–108.

First order invariant differentials have been used in the theory of uniformization and in various other problems of complex analysis. The author considers more general invariant differentials and shows that they can be employed in determining ranges of coefficients and in establishing deformation theorems for certain families of functions. Proofs are briefly indicated. Applications to the theory of Bloch's constant and to related topics are given.

L. Sario (Los Angeles, Calif.)

958:

Ikoma, Kazuo. Correction to my paper: "On Ahlfors' discs theorem and its application". Tôhoku Math. J. **9** (1957), 118.

The paper referred to appeared in the same J. **8** (1956), 101–107 [MR **18**, 567].

959:

Nilov, G. N. Proof of the fundamental theorem of higher algebra by means of the Cauchy formula. Kabardin. Gos. Ped. Inst. Uč. Zap. **8** (1955), 28. (Russian)

960:

Vekua, N. P. On a boundary value problem of linear relationship. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **24** (1957), 125–134. (Georgian. Russian summary)

The following problem is solved: find two vectors

$\phi(z)=(\phi_1, \phi_2, \cdots, \phi_n)$, $\psi(z)=(\psi_1, \psi_2, \cdots, \psi_n)$

meromorphic in a simply connected domain D^+ of the complex plane with a simple, closed, sufficiently smooth boundary L subject to the boundary conditions

$$\phi_j^+[\alpha_j(t_0)]=\sum_{k=1}^{n} A_{jk}(t_0)\psi_k^+(t_0)+\sum_{k=1}^{n} B_{jk}(t_0)\overline{\psi_k^+(t_0)}+b_j(t_0),$$

$(j=1, 2, \cdots, n)$,

where $\alpha_j(\alpha_j' \neq 0)$, A_{jk}, B_{jk}, b_j are given functions satisfying a Hölder condition and the $\phi_j^+(t_0)$ are the boundary values of $\phi_j(z)$ on L, with $\det |A_{jk}(t_0)| \neq 0$ on L. Compare the problem for one vector treated by the author in Dokl. Akad. Nauk SSSR (N.S.) **94** (1954), 173–176 [MR **15**, 785].

961:

Mesis, A. V. On Riemann's boundary problem over a field of algebraic functions for a system of n pairs of functions. Ukrain. Mat. Ž. **8** (1956), 441–449. (Russian)

962:

Bicadze, A. V. On an elementary method of solving certain boundary problems in the theory of holomorphic functions and certain singular integral equations connected with them. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 5(77), 185–190. (Russian)

The method applies to problems in which a boundary condition is prescribed along a portion of the real axis, and consists of obtaining a representation of the condition in the form $u_x(x, 0)=-\int_0^\infty p(t)(t-x)^{-1}dt$, which formula is immediately deducible from the equation for the potential of a single layer whenever the real function $p(t)$ satisfies a Hölder condition. Having this representation, one then obtains the function sought by using the Poisson-Schwarz formula $F(z)=(\pi i)^{-1}\int_{-\infty}^{\infty} u(t)\ (t-z)^{-1}dt+ib$. The author illustrates the method by solving three problems, of which a typical one is the following: to determine in the complex z-plane a function $u(x, y)$ continuous in the entire plane, behaving like $\ln|z|$ at infinity, harmonic outside the segment $0 \leqq x \leqq 1$, and assuming on this segment the given value $u(x, 0)=q(x)$.

R. N. Goss (San Diego, Calif.)

963:

Hvedelidze, B. V. On the discontinuous problem of Riemann-Privalov for several unknown functions. Soobšč. Akad. Nauk Gruzin. SSR **17** (1956), 865–872. (Russian)

The paper is concerned with a generalized Riemann problem: To a given matrix $a(t)$ and a vector $b(t)$ find an analytic vector $\phi(t)$ satisfying, along a curve Γ, (*) $\phi^+(t)=a(t)\phi^-(t)+b(t)$, where $\phi^\pm(t)$ are the respective boundary values of $\phi(t)$ along Γ. The main extension of the existing theory is the result: If $a(t)$ is a non-singular matrix, defined for t on a line Γ, satisfying a Hölder condition in a finite number of open segments of Γ, a Lipschitz condition near the endpoints c_k of these segments, not necessarily continuous at the c_k's; and $b(t)$ a vector, satisfying a Hölder condition inside the segments and, at c_k, $\lim|t-c_k|^\varepsilon b(t)=0$ for all $\varepsilon>0$ and all k; then every solution of this Riemann problem that is representable in the form of a Cauchy integral

$$(**)\quad \phi(z)=\frac{X(z)}{2\pi i}\int_\Gamma \frac{[X^+(t)]^{-1}b(t)}{t-z}\,dt+X(z)P(z),$$

where $X(z)$ is an analytic matrix satisfying $a(t)=X^+(t)[X^-(t)]^{-1}$ and $P(z)$ is entire, is a sectionwise holomorphic function.

In order that $\phi^\pm(z)$ be sectionwise continuous and $\phi(t)$ satisfy (*) and be representable in the form (**), it is necessary and sufficient that in the neighborhood of all c_k, $\|\phi(z)\| \leqq \text{const}/|z-c_k|^{\beta_k}$, $\beta_k<1$.

František Wolf (Berkeley, Calif.)

964:

Rogožin, V. S. A class of infinite systems of linear algebraic equations. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 486–489. (Russian)

The author considers systems of equations of the forms

$$(1)\quad x_n=\sum_{k=-\infty}^{\infty} a_{n-k}x_k+b_n\ (n=0, \pm1, \pm2, \cdots),$$

$$(2)\quad x_n=\sum_{k=0}^{\infty} a_{n-k}x_k+b_n\ (n=0, 1, 2, \cdots).$$

He treats these by the method of generating functions. E.g., for (2), suppose that

$$A(z)=\sum_{n=-\infty}^{\infty} a_n z^n$$

is regular for $r<|z|<R$, and that we are seeking solutions (x_n) such that $(\limsup |x_n|^{1/n})^{-1}=R_0>r$. In the homogeneous case ($b_n=0$), the problem is thus reduced to that of finding a function $X^+(z)$ regular in $|z|<R$, except perhaps for poles at the zeros of $A(z)-1$ (of orders limited by the orders of the zeros) and a function $X^-(z)$, regular in $|z|>r$ and vanishing at ∞, such that

$$X^+(z)[A(z)-1]=X^-(z)\quad (r<|z|<R).$$

This is equivalent to a Riemann problem on the circle $|z|=r+\varepsilon$, provided that ε is so small that $A(z)-1$ has no zeros in $r<|z|<r+\varepsilon$. In the non-homogeneous case, it is assumed that $(\limsup |b_n|^{1/n})^{-1}=R_1 \geqq R$; the problem to be solved then becomes

$$X^+(z)[A(z)-1]=X^-(z)+B(z)\ \ (r<|z|<R).$$

The author criticises the treatment given by Ya. N. Fel'd [same Dokl. (N.S.) **102** (1955), 257–260; MR **17**, 45] as incomplete, indicating that it may miss some solutions of the system of equations.

F. Smithies (Cambridge, England)

965:

***Schiffer, M. M. Applications of variational methods in the theory of conformal mapping.** Calculus of variations and its applications. Proceedings of Symposia in Applied Mathematics, Vol. 8, pp. 93–113. For the American Mathematical Society: McGraw-Hill Book Co., Inc., New York-Toronto-London, 1958. v+153 pp. $7.50.

The author reviews aspects of his method of interior variation for the solution of extremal problems in conformal mapping and potential theory. In addition to some familiar examples, he presents a new application to the eigenvalues of the classical Fredholm integral equation for the solution of Dirichlet's problem in the plane. For a domain with boundary curve C, let λ be the lowest eigenvalue of this integral equation corresponding to a nonconstant eigenfunction, and suppose that C is the image of the unit circle $|z|=1$ under a conformal transformation that is regular and univalent in the ring $r \leqq |z| \leqq R$, with $r<1<R$. Then it is shown that $\lambda \geqq (r^2+R^2)/(1+r^2R^2)$, and the extremal slit domains for which the equality sign holds are exhibited in a discussion based on the variational method.

P. R. Garabedian (Stanford, Calif.)

966:

Pirl, Udo. Über isotherme Kurvenscharen vorgegebenen topologischen Verlaufes und ein zugehöriges Extremalproblem der konformen Abbildung. Math. Ann. **133** (1957), 91–117.

The author considers the following extremal problem. Let B_n be a plane domain of connectivity $n(\geqq 3)$ (with non-degenerate boundary continua) and let F_i, $i=1, \cdots, k$, be non-overlapping doubly-connected domains lying in B_n, not homotopic to zero in it and no two having the same topological situation. The author shows that the maximal possible number of such domains is $2n-3$. Of these, n are homotopic to the respective boundary continua and are called boundary strips; the other $n-3$ each separate the boundary continua into two sets each with at least two elements and are called intermediate strips. The module of F_i is denoted by M_i, where this is the ratio (>1) of the radii of a conformally equivalent circular ring. Let α_i be positive real numbers. The problem is then to maximize the product $\prod_{i=1}^{2n-3} M_i^{\alpha_i}$.

The author endeavors to solve the problem by a continuity method. By boundary identification of doubly-connected domains he constructs a class of domains in which the solution of the extremal problem is fairly evident. It is then to be shown that these constitute the full class of domains in question. Although there is no doubt that the general form of solution is as indicated (it is included in a more general problem solved by the reviewer [Ann. of Math. (2) **66** (1957), 440–453; MR **19**, 845]), and it seems fairly reasonable that a solution can be constructed using a continuity method, what is required for that is a very sharp precision in the prescriptions, particularly in defining what domains make up the manifold of alleged extremal configurations. Unfortunately, as in an earlier paper on a more special problem [Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **4** (1955), 1225–1251; MR **17**, 835], the treatment seems vague on certain important points. One minor point of confusion is in the numbering as between boundary and intermediate strips which seems to be different on p. 93 and p. 96. More serious is the failure to make a quite clear prescription of what are the domains making up the manifold of extremal configurations. If the reviewer's interpretation is correct, those given fail to provide all possible extremal domains but rather constitute an open submanifold. Then the proof of completeness fails in omitting various degeneration possibilities in part 4 (p. 107). {The author, as in his previous paper, does not indicate the essential role played by quadratic differentials in the solution of this problem. In this connection the use of the techniques of the reviewer (loc. cit.) would greatly simplify the considerations of pp. 99–102. Also the reviewer would guess that with proper prescription of the extremal configurations a rather elegant form would be given to the continuity argument by using quasiconformal mappings as he did in a rather similar connection [Ann. of Math. (2) **65** (1957), 179–196; MR **18**, 568].}

After his treatment of the main problem the author indicates various modifications leading to similar problems. In particular, he mentions the case where certain of the α_i are allowed to be zero (which is equivalent to using less than the maximal number of competing doubly-connected domains) and the cases where certain boundary components degenerate to points, the corresponding α_i either equal to zero or not (in the latter case the corresponding module must be replaced by an inner conform radius). Finally he remarks that the problem can be extended to closed or finite Riemann surfaces of higher genus, allowing in addition to doubly-connected domains also quadrangles with a pair of opposite sides on boundary components. This would be the problem in the same generality as treated by the reviewer (loc. cit.). The author states that the methods of this paper apply.

J. A. Jenkins (Princeton, N.J.)

967:

Kusunoki, Yukio. Notes on meromorphic covariants. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **30** (1957), 243–249.

Starting from R. Nevanlinna's theory of Abelian integrals [Ann. Acad. Sci. Fenn. Ser. A.I. no. **100** (1951); MR **13**, 644], the author establishes Nevanlinna's first fundamental theorem for meromorphic covariants of order n on open Riemann surfaces. A counterpart of Shimizu-Ahlfors' theorem, together with other applications, is also given. *L. Sario* (Los Angeles, Calif.)

968:

Osserman, Robert. Riemann surfaces of class A. Trans. Amer. Math. Soc. **82** (1956), 217–245.

Durch die Gleichung $z=f(x, y)$, $-\infty<x<\infty$, $-\infty<y<\infty$, wird eine "über der ganzen (x, y)-Ebene liegende" Fläche des euklidischen Raumes definiert, die durch Auszeichnen der isothermen Parameter zu einer Riemannschen Fläche wird. Daß Flächen dieser Art vom parabolischen Typus sein können, ist trivial. Gibt es auch hyperbolische Flächen von dieser Art? Durch ein spezielles Verfahren, Riemannsche Flächen als gewöhnliche Flächen im euklidischen Raum zu realisieren, konstruiert Verf. Flächen vom hyperbolischen Typus und von der Form $z=f(x, y)$ [vgl. R. Osserman, Proc. Internat. Congress Math., 1954, Amsterdam, vol. 2, Noordhoff, Groningen, 1954, p. 153]. Unabhängig von dieser Frage studiert Verf. zunächst die Klasse A von einfachzusammenhängenden Riemannschen Flächen R, die als Überlagerungsflächen der z-Ebene gegeben sind und nur über dem unendlich fernen Punkt gelegene Randstellen besitzen. Man kann eine solche Fläche konstruieren durch Verheften von abzählbar unendlich vielen Blättern, die aus der Ebene durch solche geradlinige Einschnitte gegen den Nullpunkt entstehen, daß jeder Kreis nur endlich viele dieser Einschnitte trifft und genau einer dieser Einschnitte, d.i. der Fundamentaleinschnitt, bis zum Nullpunkt führt. Durch die Grundfläche wird auf R eine euklidische Metrik definiert, die allerdings in den Windungspunkten von R Singularitäten besitzt. Verf. beweist folgenden Einbettungssatz: Wenn die Riemannsche Fläche von der Klasse A ist und in jedem Blatt der Fundamentaleinschnitt frei ist, d.h. ein diesen Schnitt enthaltendes Winkelgebiet keinen andern Einschnitt enthält, und jeder Einschnitt ein (in entsprechendem Sinne) freies Ufer hat, so kann R isometrisch als Fläche von der Form $z=f(x, y)$ im euklidischen Raum realisiert werden. Um eine hyperbolische Fläche von dieser Art zu gewinnen, braucht man also nur eine Fläche der Klasse A zu konstruieren, die den Voraussetzungen des eben genannten Satzes genügt und die vom hyperbolischen Typus ist; denn es bliebt der Typus bei einer isometrischen Abbildung invariant. Die Windungspunkte von R, d.s. die Singularitäten der euklidischen Metrik auf R, erfordern allerdings noch eine besondere Betrachtung. Um den hyperbolischen Typus der Fläche festzustellen, benützt Verf. ein Kriterium von D. B. Potyagaĭlo [Ukrain. Mat. Ž. **5** (1953), 459–463; MR **15**, 787]. Es ist schade, daß sich Verf. bei seinem Kriterium für hyperbolischen Typus in Nr. 6.2 auf eine so enge

Flächenklasse beschränkt hat, die nicht einmal den Fall von Potyagaïlo umfaßt, was sich nach Ansicht des Ref. hätte bewerkstelligen lassen. Ist R eine Fläche der Klasse A und $n(t)$ die Zahl der Windungspunkte, mit Vielfachheit gezählt, die von einem festen Punkt $O \in R$ im Sinne der auf R eingeführten Metrik um weniger als t entfernt sind, und $\int^{\infty} (tn(t))^{-1}dt=\infty$, so ist R vom parabolischen Typus [Ahlfors, Comment. Math. Helv. 3 (1931), 173–177]. Von diesem Kriterium gibt Verf. folgende Verschärfung: Wenn die Projektion der Windungspunkte von R auf die Grundfläche beschränkt ist, so genügt schon die Divergenz des Integrals $\int^{\infty} (n(t))^{-1}dt$ für den parabolischen Typus. Es wird dann weiter eine Teilklasse O von Flächen der Klasse A angegeben, in welcher die Bedingung $\int^{\infty} (tn(t))^{-1}dt=\infty$ hinreichend und notwendig ist für den parabolischen Typus. Für jedes $\varepsilon>0$ gibt es in dieser Klasse O Flächen vom hyperbolischen Typus mit $n(t)\leqq t^{\varepsilon}$.

A. Pfluger (Zbl 71 (1957), 74)

969:

Hölder, E. Fortsetzung Abelscher Differentiale 1. Gattung ins Nichtlineare. Ann. Acad. Sci. Fenn. Ser. A. I. no. **250/15** (1958), 5 pp.

The author begins by stating a specialization, for arbitrary closed differential forms on an arbitrary compact Riemann surface, of a general integral formula previously developed by him. Using that formalism, he proves the existence and uniqueness of the velocity distribution for a stationary isentropic potential flow of a gas on such a Riemann surface, provided that the periods of the differential form $\pi=udx+vdy$ (for the velocity vector) are sufficiently small. The formalism enables the non-linear differential equations for such a flow to be put in the form of a non-linear integral equation for the complex differential form $\pi+i*\pi$, $*\pi$ being the dual of π.

C. B. Morrey, Jr. (Berkeley, Calif.)

970:

Huber, Alfred. On subharmonic functions and differential geometry in the large. Comment. Math. Helv. 32 (1957), 13–72.

Let the metric of an open, two-dimensional Riemannian manifold M be defined, in terms of local parameters, by a positive definite quadratic form having coefficients of class C^3. (As the author notes, the results presented in the paper here reviewed actually are valid under the weaker hypothesis that the coefficients are of class C^1 and that M has a continuous Gaussian curvature in the sense of H. Weyl.) Introducing local isothermic parameters, one expresses the metric by $ds^2=e^{2u(z)}|dz|^2$, $z=x+iy$. In accordance with the theorema egregium, the Gaussian curvature K of M can be expressed in terms of the above-mentioned coefficients and their derivatives of the first and second orders; for an isothermic system, the expression is simply $K=-e^{-2u}\Delta u$, $\Delta\equiv\partial^2/\partial x^2+\partial^2/\partial y^2$. Hence, letting $dA=e^{2u}dxdy$ denote the element of area on M, one has $KdA=-\Delta udxdy$. In general, isothermic parameters can be introduced only in the small, so that in order to treat problems pertaining to differential geometry in the large, one considers the Riemann surface S determined by the conformal structure of M; if M is not orientable, then it is simply replaced by an orientable, two-sheeted covering surface. The local uniformizers are now defined as functions that map portions of M conformally onto regions in the z-plane, and M is conceived of as a Riemann surface on which a conformal metric $ds=e^{u(z)}|dz|$ has been introduced in terms of the local uniformizer z.

The fact that the foregoing formula for K involves the Laplacian of u indicates a connection between differential geometry and potential theory. Thus, u is subharmonic (superharmonic) in a certain parameter region if and only if $K\leqq 0$ ($\geqq 0$) on the corresponding portion of M; again, the surface integral of K, considered as a set function, is essentially the measure associated with u (i.e., the mass distribution of density $\Delta u/(2\pi)$ that appears when w is represented as a sum of a logarithmic potential and a harmonic function). Accordingly, results in differential geometry in the large involving the curvatura integra have a potential-theoretic interpretation; the present paper explores this point of view.

The author makes the mild assumption that the function $u(z)$ can be expressed as the difference of two subharmonic functions, $u(z)=u_1(z)-u_2(z)$; such a representation always exists, for example, when u is of class C^2. Though neither the functions u_1 and u_2 nor their associated measures μ_1 and μ_2 are uniquely determined, the difference $\mu=\mu_1-\mu_2$ is independent both of the uniformizer and of the decomposition. Let $\mu=\mu^+-\mu^-$ denote the Jordan decomposition of μ, so that $\mu^+(e)\leqq\mu_1(e)$ and $\mu^-(e)\leqq\mu_2(e)$ for all Borel sets e and for all decompositions $\mu=\mu_1-\mu_2$ of μ as a difference of positive measures, and let $C^+=2\pi\mu^-(S)$ and $C^-=2\pi\mu^+(S)$. Then the difference $C=C^+-C^-=-2\pi\mu(S)$, defined whenever C^+ and C^- are not both infinite, is called the curvatura integra of the metric, since for sufficiently regular functions $u(z)$ one has

$$C^+=\iint_S \Delta^- u\,dxdy=\iint_M K^+dA,$$

$$C^-=\iint_S \Delta^+ u\,dxdy=\iint_M K^-dA,$$

$$C=-\iint_S \Delta u\,dxdy=\iint_M KdA,$$

where $\Delta^+u=\max[\Delta u, 0]$, $\Delta^-u=\max[-\Delta u, 0]$, $K^+=\max[K, 0]$, and $K^-=\max[-K, 0]$. The metric $u(z)$ is said to be complete on an open Riemann surface S if $\int_\sigma e^{u(z)}|dz|=+\infty$ for every locally rectifiable path σ that tends to the ideal boundary of S.

After an introductory first section, in the second section the author uses subharmonic-function methods to develop several theorems, needed for subsequent applications, concerning conformal metrics defined in schlicht annular regions. The following isolated application is given in the third section: An entire analytic function $w=f(z)$ is a polynomial if and only if there exists a positive number λ such that

$$\int_\sigma |f(z)|^{-\lambda}|dz|=+\infty$$

for every locally rectifiable path σ tending to infinity. The author leaves open the question as to whether $\int_1^{+\infty} |f(\rho e^{i\theta})|^{-\lambda}d\rho=+\infty$ for all θ, $0\leqq\theta<2\pi$, implies that $f(z)$ is a polynomial.

The remaining three sections are devoted to geometric results, including the following.

Let S be a finitely connected, open Riemann surface on which a complete conformal metric $e^{u(z)}|dz|$ is defined. Suppose that the curvatura integra C exists. Then C satisfies the inequality $C\leqq 2\pi\chi$, where χ denotes the Euler-Poincaré characteristic of S. If, in addition, S is of finite total area $A=\iint_S e^{2u}dx\,dy$, then the sign of equality holds: $C=2\pi\chi$.

If the conformal metric $e^{u(z)}|dz|$, defined on an infinitely connected Riemann surface S, is complete, then $C^-=+\infty$.

Let S be an open Riemann surface on which a complete

conformal metric $e^{u(z)}|dz|$ is defined. Suppose that the measure μ^+ has a compact support. Then the total area $A=\iint_S e^{2u}dxdy$ of S is infinite.

If an open Riemann surface S admits a complete conformal metric $e^{u(z)}|dz|$ with finite C^-, then S is of parabolic type. *E. F. Beckenbach* (Los Angeles, Calif.)

971:

Ahiezer, N. I.; and Levin, B. Ya. Inequalities for derivatives analogous to Bernstein's inequality. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 735–738. (Russian)

The authors announce a general theorem which, in particular, gives a precise bound for the derivative of an entire function of exponential type on a set when a bound is prescribed for the function on the same set. Let E be a perfect set on the real axis, having positive harmonic measure, and let G denote the z plane with E deleted. Let a region be called of type A, B, C, respectively, if it is an upper half plane, an upper quadrant, or an upper half strip, in each case with a finite or infinite number of parallel finite linear cuts starting on its base, and having limit lines only on the remaining boundary (if any). Theorem 1: The upper half plane can always be mapped conformally on a region Δ of one of these types so that E becomes the base of the region, i.e., the part of its boundary that belongs to the real axis. Let the mapping function be extended to G by symmetry, becoming a (multiple-valued) function ϕ, whose imaginary part v is single-valued, positive in G and zero on its boundary. Let $\omega(z)=e^{-i\phi(z)}$. In G, consider an analytic, possibly multiple-valued, function f, and let $f^*(z)=\sup|f(z)|$. The degree of f is $\limsup |z|^{-1}\log f^*(z)$ as $|z|\to\infty$, $z\in G$. Theorem 2: Let f have finite degree in G; (a) let it approach a limit of modulus at most 1 at each point of E; and (b) for each $\varepsilon>0$ let there be a $\delta>0$ such that $f^*(z)[\omega(z)]^{-\sigma-\varepsilon}\to 0$ (for some nonnegative σ) as $|z|\to\infty$, $|\arg z\pm\frac{1}{2}\pi|\leqq\delta$. Then $|f(z)|\leqq|\omega(z)|^\sigma$ in G, with equality only if $f(z)=e^{i\gamma}[\omega(z)]^\sigma$. Now say that $f\in K_\sigma$ if f is of finite degree in G, satisfies (a) of Theorem 2, and is a linear combination of two functions each of which satisfies (b) and is real on E. Theorem 3: If $f\in K_\sigma$, with $\sigma>0$ if Δ is of type A or B, $\sigma\geqq\pi/\lambda$ if Δ is of type C and of width λ; then at every point of E where $f'(x)$ and $\omega'(x)$ exist, $|f'(x)|\leqq\sigma|\omega'(x)|$, with equality for a linear combination of $\omega(z)^\sigma$ and $\omega(z)^{-\sigma}$. If E consists only of segments, all extremal functions are of this form. This includes Schaeffer's theorem [Duke Math. J. **20** (1953), 77–88; MR **14**, 631]. In special cases the bound is obtained explicitly: (1) E is $[-1, 1]$; $|f'(x)|\leqq\sigma(1-x^2)^{-\frac{1}{2}}$. (2) E is $[0, \infty)$; $|f'(x)|\leqq\frac{1}{2}\sigma x^{-\frac{1}{2}}$. (3) E is $(-\infty, -1]\cup[1, \infty)$; $|f'(x)|\leqq\sigma|x/\sqrt{(x^2-1)}|$; from this the authors deduce (4) an analogue of Markov's theorem, namely, that if $\sigma\geqq\sqrt{3}$, $g(z)$ is entire and of exponential type σ, and $\sup_E|g(x)|\leqq 1$, then $|g'(x)|\leqq\sigma^2$ in E. (5) E is $[-1, \beta]\cup[\alpha, 1]$. (6) E is $(-\infty, -1]\cup[\beta, \alpha]\cup[1, \infty)$. *R. P. Boas, Jr.* (Evanston, Ill.)

972:

Boas, R. P., Jr.; and Schaeffer, A. C. Variational methods in entire functions. Amer. J. Math. **79** (1957), 857–884.

The principal result of this paper is as follows. Let F_τ be the set of entire functions of exponential type $\leqq\tau$ and bounded by 1 in absolute value on the real axis. Let

$$L(g)=\sum_{\nu=1}^{m}\sum_{j=0}^{n_\nu}\alpha_{\nu j}g^{(j)}(x_\nu),$$

where the x_ν are m distinct real numbers, the $\alpha_{\nu j}$ are real and at least one $n_\nu>0$. Then $\sup|L(g)|$ ($g\in F_\tau$) is attained for a function $f(z)$ of order one and type exactly equal to τ. This function satisfies a differential equation of the form $(f'(z))^2/(1-f^2(z))=\tau^2p^2(z)/q(z)$, where p and q are polynomials subject to certain simple conditions. This result is applied to the determination of $\sup|L(g)|$ in certain cases, notably for $L(g)=f'(\frac{1}{2})+f'(-\frac{1}{2})$. The proof of the principal theorem is based on a direct investigation of the change in $L(g)$ when g is replaced by $g+\varepsilon g_1$, where g_1 runs through suitably selected functions in F_τ.

W. H. J. Fuchs (Ithaca, N.Y.)

973:

Zaĭcev, M. N. On the coefficients of entire analytic functions. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **12** (1957), no. 3, 3–8. (Russian)

Let $F(z)=\sum a_nz^n$ be an entire function of order ρ_0 greater than 1, and let γ be the density of nonzero a_n. Let $F(z)$ satisfy the differential equation (of infinite order) $\sum c_nF^{(n)}(z)=0$, where $\phi(t)=\sum c_nt^n$ is of exponential type, with simple zeros. The author shows that if there are disks centered at the zeros of φ in which $|\varphi(t)|>\exp\{-|t|(\rho-\delta)/(\rho-1)\}$, where $\rho>\rho_0$, $0<\delta<1$, then $\gamma\geqq 1/(2\rho_0)$. This generalizes a result by Gelfond, in which $F(z)$ was periodic [Izv. Akad. Nauk SSSR. Ser. Mat. **5** (1941), 95–98; MR **2**, 356]. *R. P. Boas, Jr.* (Evanston, Ill.)

974:

Singh, S. K. A note on entire and meromorphic functions. Proc. Amer. Math. Soc. **9** (1958), 6–10.

The author gives elementary proofs of three known theorems. (1) There is at most one entire function g such that $\log M(r, g)=o(\log M(r, f))$ and the exponent of convergence of the zeros of $f-g$ is less than the order of f. (2) If f is an entire function of nonintegral order, then $\limsup N(r, a)/\log M(r, f)>0$ for all finite a. (3) If F is a meromorphic function of order $\rho<1$, then $\limsup\{N(r, a)+N(r, b)\}/T(r, f)\geqq 1-\rho$ for all $a\neq b$, finite or infinite. *R. P. Boas, Jr.* (Evanston, Ill.)

975:

Heins, Maurice. Asymptotic spots of entire and meromorphic functions. Ann. of Math. (2) **66** (1957), 430–439.

Etude particulière des comportements asymptotiques [pour la définition de "asymptotic spot" voir M. Heins, Ann. of Math. (2) **61** (1955), 440–473; MR **16**, 1011] pour des fonctions entières et méromorphes.

Un indice d'harmonicité est lié à chaque régionnement asymptotique par la décomposition de la plus grande minorante harmonique de la fonction de Green $G_\Omega(f_{\sigma(\Omega)}, w_0)$ [cf. Lindelöf Principle]. La somme des indices d'harmonicité est liée à la croissance de f par la fonction caractéristique $T(r, f)$. D'autres relations limitent la singularité du comportement de f lorsque le régionnement asymptotique correspond à une valeur w_0 finie.

L'auteur établit d'autre part la dénombrabilité de l'ensemble des valeurs localement non prises, pour une fonction méromorphe dans le plan, et montre que c'est la seule restriction à imposer à cet ensemble.

L. Fourès (Marseille)

976:

de La Vallée Poussin, C. Fonctions périodiques douées de valeurs exceptionnelles. Structure. Théorème de Picard. Ann. Soc. Sci. Bruxelles. Sér. I. **71** (1957), 73–88.

General remarks on exceptional and asymptotic values of analytic functions, based on topological con-

siderations. Vague formulations may account for seeming contradictions with known facts.

L. Ahlfors (Cambridge, Mass.)

977:

Kuz'mina, G. V. Determination of the least radius of univalency for a certain class of analytical functions. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 751–754. (Russian)

For the class of functions $H_1^*(a)$ such that $f(z,a)=\sum_{k=0}^{\infty} c_k z^k=[F(z,a)]^2=(\sum_{k=0}^{\infty} d_k z^k)^2$, where $F(z)$ is regular for $|z|<1$, $\sum_{k=0}^{\infty}|d_k|^2=1$, and $0<|c_1|=2|d_0||d_1|=a\leq 1$, the author determines the radius of univalency $R=\min(r_1, r_2)$, where r_1 and r_2 are the least positive roots of the equations $a^2-2ar+2ar^3-r^4=0$, $a^2-(4-a^2)r^2+3r^4-r^6=0$ respectively. Extremal functions are given without examining their uniqueness.

V. Linis (Ottawa, Ont.)

978:

Aleksandrov, I. A. On bounds for convexity and starlikeness for functions univalent and regular in a circle. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 903–905. (Russian)

The exact bound for convexity for functions of the class S (regular and univalent for $|z|<1$, $f=z+a_2z^2+\cdots$) was determined by R. Nevanlinna [Översikt Finska Vet. Soc. Förh. **62A** (1919–1920), no. 7] as $R_c=2-\sqrt{3}$, and the exact bound for starlikeness by H. Grunsky [Jber. Deutsch. Math. Verein. **43** (1933), 140–143] as $R_s=\text{th}\frac{1}{4}\pi$. The author generalizes these results in several ways: a circle with centre at ω, $|\omega|<1$ and radius $\leq 2-(3+|\omega|^2)^{\frac{1}{2}}$ is mapped on a convex domain; every noneuclidean circle with centre at ω and of noneuclidean radius $\leq\frac{1}{2}\pi$ is mapped on a domain starlike with respect to $f(\omega)$. Bounds are exact. The author also considers bounds of starlikeness with respect to an arbitrary point in the image domain, and in the case of generalized starlikeness as defined by G. M. Golusin [Mat. Sb. **42** (1935), 169–190].

V. Linis (Ottawa, Ont.)

979:

Sario, Leo. Strong and weak boundary components. J. Analyse Math. **5** (1956/57), 389–398.

Let W be a plane region and γ a boundary component of W. If γ remains a point (or proper continuum) under every schlicht conformal mapping of W, it is called weak (or strong, resp.). In the first two theorems the author gives criteria for γ to be weak and strong, respectively. Next we assume that W is bounded by an analytic Jordan curve γ and a closed set δ surrounded by γ, with $\gamma\cap\delta=\emptyset$. For $\zeta\in W$, we consider the class $\mathfrak{F}$ of schlicht analytic functions on W with the property that $F(\zeta)=0$, $F'(\zeta)=1$, $|F(z)|=\text{const}\ r(F)$ on γ. We enclose δ by a finite number of analytic Jordan curves δ_Ω such that γ and δ_Ω bound a region Ω. The logarithmic area $A(F)$ enclosed by δ is defined by $\lim_{\Omega\to W}-\int_{\delta_\Omega}\log|F(z)|d\arg F(z)$. In Theorem 3 it is proved that there exists a unique function $F_0(z)$ in $\mathfrak{F}$ which gives $\max_{F\in\mathfrak{F}}(2\pi\log r(F)+A(F))=2\pi\log r(F_0)$, that the deviation from the maximum is equal to the Dirichlet integral $D(\log|F/F_0|)$ for any $F\in\mathfrak{F}$, that the area of the image δ_{F_0} under F_0 of δ vanishes and that each component of δ_{F_0} is a radial slit or a point. Corresponding results are given to the minimizing problem of $2\pi\log r(F)-A(F)$. Finally Theorem 3 is applied to prove that the value $r(F)$ does not depend on $F\in\mathfrak{F}$, if and only if the complement of δ with respect to the whole plane does not bear any nonconstant analytic function with a finite Dirichlet integral.

M. Ohtsuka (Hiroshima)

980:

Hornich, Hans. Zur Struktur der schlichten Funktionen. Abh. Math. Sem. Univ. Hamburg **22** (1958), 38–49.

The author considers the space $\mathfrak{R}$ of all functions $f(z)=\sum_1^\infty a_n z^n$ which are regular in $|z|<1$ and vanish at the origin. He shows that a function $f\in\mathfrak{R}$ is univalent if and only if $f(\phi,z)\equiv\sum a_n(\sin n\phi/\sin\phi)z^{n-1}\neq 0$ for real ϕ and $|z|<1$. These functions $f(\phi,z)$, which are regular in $|z|<1$, are called the functions associated with $f(z)$. The author introduces a metric in $\mathfrak{R}$ by defining the distance between two functions f_1 and f_2 as $\|f_1-f_2\|$, where $\|f\|=\sup|a_n|^{1/n}$. He then proves the following propositions concerning this metric space $\mathfrak{R}$.

If S is the subset of the univalent functions of $\mathfrak{R}$ (and the function $f\equiv 0$) then S is closed and $T=\mathfrak{R}-S$ is open. A function $f\in S$ is called "stably" or "unstably univalent" according as f is an interior or a boundary point of S. Each of the following two conditions is necessary and sufficient for stable univalence of $f(z)$: (i) $|f(\phi,z)|\geq\eta>0$ for real ϕ and $|z|<1$; (ii) $|f(z_1)-f(z_2)|/|z_1-z_2|\geq h>0$ for $z_1\neq z_2$, $|z_j|<1$.

Let $\mathfrak{R}$ be divided into its connected components K. Two functions f_1 and f_2 belong to the same component K if and only if f_1-f_2 is an entire function. Each two components have a positive distance from one another, and the cardinal number of the set of all components K is that of the continuum. Finally, in each component K the subset $K\cap T$ of non-univalent functions is itself connected; and in the component K_0, which consists of all entire functions of $\mathfrak{R}$, the subset of stably univalent functions is also connected.

{It may be noted that in § 2 the proof that T is open can be simplified by the application of Rouché's theorem to the functions $f(\phi_0,z)$ and $g(\phi_0,z)$.}

F. Herzog (East Lansing, Mich.)

981:

Gel'fer, S. A. On coefficients of typically real functions. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 211–213. (Russian)

The function $f(z)$ is said to be typically-real in the unit circle if it is regular there and if, for z not real, $\Im f(z)\Im z>0$. The normalized class with $f(0)=0$ and $f'(0)=1$ is denoted by T. The author studies the subclass $T^{(2)}$ of functions for which $f(z)\in T$ and also $f(iz)/i\in T$. It is known that such functions are odd. Set $f(z)=z+\sum_{k=1}^{\infty}x_{2k}z^{2k+1}$. The author proves that: if $f(z)\in T^{(2)}$, then $-(1+3^{\frac{1}{2}}/18)\leq x_3\leq 1+3^{\frac{1}{2}}/18$; $x_{2k}\leq 3/2$, $k=1, 2, \cdots$; $x_4\geq -2/3$; and $x_6\geq -5/16-121/48\cdot 3^{\frac{1}{2}}$; and each of these inequalities is sharp, equality occurring for a unique function of the class $T^{(2)}$.

A. W. Goodman (Lexington, Ky.)

982:

Kas'yanyuk, S. A. On functions of classes A and $\mathfrak{H}_\delta$ on an annulus. Mat. Sb. N.S. **42(84)** (1957), 301–326. (Russian)

Let c_j ($j=1, 2, \cdots, n$) be a set of points in the circle $|z|<R$. About the points c_j construct nonoverlapping circles of radii r_j such that $|c_j|+r_j<R$. Let $K_z[c_j; r_j; R]$ be the region with boundary $|z|=R$ and $|z-c_j|=r_j$ ($j=1, 2, \cdots, n$). The author considers the functions $H(z,a)$, called primitive Blaschke functions, which vanish at $z=a$ and satisfy (i) $|H(Re^{i\theta},a)|=1$, (ii) $|H(c_j+r_je^{i\theta},a)|=P_j<1$, (iii) $|H(z,a)|\leq 1$ for $z\in K_z[c_j; r_j; R]$. Using this function he constructs functions which are analytic in $K_z[c_j; r_j; R]$ with moduli not exceeding unity and having zeros at prescribed sequences of points having limit

points on the circles $|z-c_j|=r_j$. This generalizes results given in a monograph by Privalov [Boundary properties of analytic functions, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950; MR 13, 926].

A function $f(z)$ analytic in the ring $K_z[r; R]$ $(r<|z|<R)$ is said to belong to Class A if $\int_{-\pi}^{\pi} \ln^+|f(\rho e^{i\theta})|d\theta<K_A(f)<\infty$ for $r<\rho<R$, where as usual $\ln^+ a=\ln a$ if $a\geqq 1$ and $\ln^+ a=0$ if $a<1$. A function $\phi(z)$ analytic in $K_z[r; R]$ is said to belong to Class $\mathfrak{H}_\delta$ if $(2\pi)^{-1}\int_{-\pi}^{\pi}|\phi(\rho e^{i\theta})|^\delta d\theta<K_H(\phi)$, $r<\rho<R$. Employing the properties of the function $H(z, a)$, the author to is able show that if $f(z)$ belongs to Class A then $f(z)=b(z)\Phi(z)$, where $b(z)$ is a Blaschke function and $\Phi(z)$ has no zeros in $K_z[r; R]$. A similar theorem holds for functions belonging to Class $\mathfrak{H}_\delta$. From these results and structure formulas developed by Zmorovič [Mat. Sb. N.S. **40(82)** (1956), 225–238; MR **18**, 648] for the class of functions $\psi(z)$ analytic in $K_z[r; R]$ and satisfying $(2\pi)^{-1}\int_{-\pi}^{\pi}|\Re\{\psi(z)\}|d\theta<L$, a structure formula for functions belonging to Class A is found. The rate of growth of the modulus of functions belonging to Class A is studied.

W. C. Royster (Lexington, Ky.)

983:

***Leont'ev, A. F. On properties of sequences of linear aggregates that converge in a region in which the system of functions generating the linear aggregates is not complete.** American Mathematical Society Translations, Ser. 2, Vol. 10, pp. 1–12. American Mathematical Society, Providence, R. I., 1958. iv+409 pp. $6.60.

Translated from Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 26–37; MR **18**, 728.

984:

***Boas, Ralph P., Jr.; and Buck, R. Creighton. Polynomial expansions of analytic functions.** Ergebnisse der Mathematik und ihrer Grenzgebiete. Neue Folge. Heft 19. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. viii+77 pp. DM 19.80.

The objects of this monograph are to explain a general method, the method of kernel expansion, of expanding analytic functions in series of the form

$$f(z)=\sum c_n p_n(z), \tag{1}$$

and to illustrate the power of the method by working out the details in a large number of important cases. The series in (1) is required to represent f in some specified sense, usually convergence or summability uniform on compact subsets of a region Ω_0. The mapping $(c_n)\to f$ so defined is not necessarily one-to-one, so that there may be several expansion formulae in the form of right inverses $f\to(\mathscr{L}_n(f))$, where $\mathscr{L}_n$ is a linear functional.

Two problems arise, according as the $(p_n(z))$ or the $(\mathscr{L}_n)$ are given: these are respectively the expansion problem and the interpolation problem. For the first of these, the method is as follows. Functions (u_n) are chosen so that

$$K(z, \zeta)=\sum p_n(z)u_n(\zeta) \tag{2}$$

is a suitable kernel for the purpose of expressing f in an integral transform

$$f(z)=(2\pi i)^{-1}\int_\Gamma K(z, \zeta)F(\zeta)d\zeta. \tag{3}$$

A primary requirement for suitability is that, for each compact subset C of Ω_0, Γ can be chosen so that (3) holds and also the series in (2) converges (or is summable in the relevant sense) uniformly on $C\times\Gamma$: such a Γ is said to be admissible. The expansion (1), valid in Ω_0, now follows by multiplying (2) by $F(\zeta)$ and integrating term by term. The coefficients are given by

$$c_n=\mathscr{L}_n(f)=(2\pi i)^{-1}\int_\Gamma F(\zeta)u_n(\zeta)d\zeta. \tag{4}$$

In the interpolation problem, the $\mathscr{L}_n$ being given, a kernel is chosen so that (3) holds and (4) can be solved for the (u_n). If, moreover, $(p_n(z))$ can be found to satisfy (2), we have the same equations as before and again obtain an expansion (1).

Stated with this degree of generality, there appears to be a good deal of scope in the choice of kernel. However, aside from questions of suitability, with the "wrong" kernel the transform (3) may represent only an unnecessarily small class of f, leading to a result which falls short of the full truth. Also, in the illustrative examples, a kernel is usually given with the polynomials $(p_n(z))$. Indeed, the latter are nearly all specializations of the generalised Appell polynomials, namely those given by a generating relation

$$A(w)\Psi(zg(w))=\sum p_n(z)w^n, \tag{5}$$

where $A(0)\neq 0$, $g'(0)\neq 0$ and $\Psi^{(n)}(0)\neq 0$ for any n. Putting $\zeta=g(w)$, $w=W(\zeta)$, this takes the form (2) with $K(z, \zeta)=\Psi(z\zeta)$, $u_n(\zeta)=[W(\zeta)]^n/A(W(\zeta))$.

The general theory outlined above is the subject matter of Chapter I. In Chapter II is investigated the problem of expanding entire functions. Here, Ω_0 is the entire plane and the class of f representable in the form (1) is determined in terms of the relation between the singularity set $D(f)$ of the transform F and the convergence set of (2): $D(f)$ must be such that an admissible path Γ exists. The determination of this class is couched in terms of Ψ-type, a notion which reduces to that of exponential type when $\Psi=\exp$. For functions regular at the origin (Chap. III), a region Ω is chosen and the region Ω_0 (in which every f regular in Ω is represented) is then determined by similar requirements. In both cases, results can be stated in a form suitable for application to specific sets of polynomials, and most of those hitherto treated in the literature are of generalised Appell type and can be (and are) treated here. Each polynomial set encountered is given a name, usually that of an appropriate writer. There is a section indicating how the kernel can sometimes be improved. A final short chapter gives applications to uniqueness theorems (cf. Carlson's Theorem) and functional equations (linear differential equations of infinite order; difference equations). There is an index and a bibliography not confined to titles that are referred to in the text. The production and printings are up to the usual high standard of the Ergebnisse series.

{Reviewer's Comment. Although the compromise between brevity and clarity is well-handled, there are about a dozen points which remain obscure to the reviewer. Statements concerning Whittaker's basic series are not always accurate (e.g. the proof of theorem 7.6 is valid only on the assumption that C lies in the disc of convergence of the power series for u_n, and without this assumption the result can fail). The expansions of zero given in theorems 8.4 and 8.7 are only formal and may not converge. Theorem 14.12 asserts more than is proved (or true, the set used by the authors to illustrate their theorem providing a counter-example): the region Ω must be required to contain the closed disc $|z|\leqq 1/\rho_0$. A result on p. 65 is vacuous because the hypotheses are self-contradictory (possibly due to a misquotation; the original paper [Boas, J. Indian Math. Soc. (N.S.) **14** (1950), 1–14;

MR 12, 248] was not available to the reviewer). A result of Sheffer is quoted on p. 70 with incomplete hypotheses, and on p. 71 the (trivial) case $\beta=0$ is overlooked. There are nearly two dozen misprints.}

{Editor's note. The authors write that the objection above to theorem 7.6 is apparently based on a misinterpretation, and that the difficulty with theorem 14.12 arises from the misprint $<$ for $\leqq$ in the text of the book.}

W. F. Newns (Liverpool)

985:

Falgas, Maurice. Sur les séries de base relatives à certaines classes de fonctions entières. C. R. Acad. Sci. Paris **245** (1957), 1208–1211.

This is a continuation of same C. R. **244** (1957), 2275–2278 [MR **19**, 737]. Let $\{q_n(z)\}$ be the Faber polynomials associated with a region D and set

$$(*) \qquad Q_n(z)=(2\pi i)^{-1}\int_{\partial D} q_n(\zeta)E_\rho(z/\zeta)\zeta^{-1}d\zeta,$$

where $E_\rho(t)$ is the entire function $\sum_0^\infty t^n/\Gamma(1+n/\rho)$. Then, one may define a class $\mathscr{C}_\rho$ of entire functions of order at most ρ such that any $f\in\mathscr{C}_\rho$ has a unique series expansion of the form $\sum c_nQ_n(z)$, with $\limsup |c_n|^{1/n}<1$. This can be regarded as a special case of a general theory which may deal with the transition induced by a transformation such as (*) between polynomial sets, and the relationship between their expansion properties. [See also the book reviewed above.] *R. C. Buck* (Madison, Wis.)

986:

Nassif, M. Note on convergence of the product of basic sets of polynomials. Amer. J. Math. **79** (1957), 943–948.

In a previous paper [M. Nassif Ghabbour, same J. **69** (1947), 583–591; MR **9**, 22], the author gave an invalid proof of a theorem on the effectiveness of a product set. He now proves the conclusion of that theorem on modified hypotheses. The proof uses a lemma on inverse sets, and five examples illustrate the effect of weakening the hypotheses in the theorem or in the lemma.

W. F. Newns (Liverpool)

987:

Evgrafov, M. A. Determination of the class of convergence in certain interpolation problems. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 31–33. (Russian)

Let $\{L_n\}$ be a system of linear functionals on entire functions F, defined by

$$L_n(F)=(2\pi i)^{-1}\int_C F(z)u_n(1/z)z^{-1}dz,$$

$$u_n(z)=z^n+\sum_{k=1}^{\infty}a_{n,k}z^{n+k}.$$

Let $P_n(z)$ be the polynomials biorthogonal to the functionals L_n. The author seeks sufficient conditions for the validity of the representation $F(z)=\sum L_n(F)P_n(z)$. He supposes that $u_n(z/\lambda_n)\neq 0$ (for $n>n_0(z)$) in a region E and that $u_{n+1}(z/\lambda_{n+1})/u_n(z/\lambda_n)\to u(z)$, with $u(0)=0$, $u'(0)=1$, uniformly in E; here $\lambda_n=\lambda(n)$, where: $\lambda(z)$ is regular in the z-plane cut from $-\infty$ to 0; $\lambda(z)>0$ for $z>0$; $w=\lambda(z)$ maps $\Re(z)>0$ on a region D bounded by curves whose equations near ∞ are $\arg w=\pm\frac{1}{2}\alpha(|w|)$; $\alpha(r)$ has two monotonic derivatives near $r=\infty$; $\frac{1}{2}<\lim \pi/\alpha(r)\leqq\infty$; $\lim r\alpha'(r)/\alpha(r)=0$. Let $h(z)$ be the branch of the inverse of $\lambda(z)$ which is positive for positive z. Let $\Phi(z)=(2\pi i)^{-1}\int_L(\zeta-z)^{-1}e^{h(\zeta)}d\zeta$, with L given by $\arg\zeta=\pm\frac{1}{2}\alpha(|\zeta|)$ near ∞. Let $v_n(z)=(2\pi i)^{-1}\int_C u_n(z/\zeta)\Phi(\zeta)\zeta^{-1}d\zeta$; the author shows that, under a supplementary hypothesis on the $u_n(z)$, we have $v_{n+1}(z)/v_n(z)\to v(z)$. His theorem then states that if $v(z)$ is regular and univalent in a star-shaped subset K of E which $w=v(z)$ maps on a disk, and if $F(z)$ can be represented as $F(z)=(2\pi i)^{-1}\int_C\Phi(z\zeta)f(\zeta)d\zeta$ with f regular outside K, then $F(z)=\sum L_n(F)P_n(z)$. The Abel-Gončarov case corresponds to $u_n(z)=z^n(z-\lambda_n)^{-n-1}$.

R. P. Boas, Jr. (Evanston, Ill.)

FUNCTIONS OF SEVERAL COMPLEX VARIABLES, COMPLEX MANIFOLDS

See also 1235, 1339, 1340.

988:

Michiwaki, Yoshimasa. On some property of function in several complex variables. Sci. Rep. Tokyo Kyoiku Daigaku Sect. A. **5** (1956), 283–286.

This paper contains a literal generalization from 1 to k complex variables of the following theorem on holomorphic mapping: If $w=w(z)$ is given in $\|z\|<R$ and $\|w(z)\|\leqq S\{w(0),\theta\}$ in $\|z\|\leqq R\theta(0<\theta<1)$ and $\det w'(0)\neq 0$, then $R\leqq L(w(0),w'(0),\theta)$. The author reduces this theorem to the following one of F. Bureau [J. Math. Pures Appl. (9) **31** (1952), 161–190; MR **14**, 37], for which he gives a new short proof: If $\|w(z)\|<1$ in $\|z\|<1$, then

$$\left\|\frac{dw(z)}{dz}\right\|\leqq\frac{(1-\|w(z)\|^2)^{\frac{1}{2}}}{1-\|z\|^2}.$$

S. Bochner (Princeton, N.J.)

989:

Ivanov, V. K. Relation between the growth of an entire function of several variables and the distribution of singularities of a function associated with it. Mat. Sb. N.S. **43(85)** (1957), 367–378. (Russian)

The author generalizes the Pólya representation of an entire function of exponential type to the case of several complex variables. [For other investigations in the same direction see Kozmanova, Dokl. Akad. Nauk SSSR. (N.S.) **113** (1957), 1203–1205; MR **19**, 738; Džrbašyan, Mat. Sb. N.S. **41(83)** (1957), 257–276; MR **19**, 539; Ronkin, ibid. **39(81)** (1956), 253–266; MR **18**, 569.] As in the one-dimensional case he is able to identify a set determined by the growth of the function with the set of singular points of a transform, and obtain a representation of the entire function as an integral.

R. P. Boas, Jr. (Evanston, Ill.)

990:

Wintner, Aurel. Schwarz's lemma and a singularity of Briot-Bouquet. Amer. J. Math. **79** (1957), 778–796.

The main purpose of this paper is to obtain the proof and several consequences of the following lemma: If for $|z|<1$ and $|w|<1$, the function of two complex variables $f(z,w)$ is regular and $|f(z,w)|\leqq 1$, and if $f(0,0)=0$, then the initial-value problem (*) $zw'=f(z,w)$, with $w=0$ for $z=0$, has a unique regular solution on $|z|<1$. The lemma is proved by using a modified version of the Schwarz lemma, and it is shown too that the condition $|f(z,w)|\leqq 1$ cannot be weakened and that the value 1 for the radius of the domain of existence of the solution $w(z)$ is the best possible. As a consequence, the results of Briot and Bouquet [J. Ecole Polytech., cah. **36** (1856), 133–198], concerning the equation $zw'=\alpha z+\beta w+\sum_{m+n>1}a_{mn}z^mw^n$ are sharpened. In addition, it is also proved that the existence theorem for implicit functions that results by replacing (*) by $w=f(z,w)$ in the statement of the lemma,

is valid as well. Furthermore, there are a number of comments and improvements to previous and classical results. *A. G. Azpeitia* (Amherst, Mass.)

991a:

Pyateckiĭ-Šapiro, I. I. Singular modular functions. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 53–98. (Russian)

In this paper is given a generalization of the abelian modular functions introduced by Siegel.

The Riemann-Frobenius condition on the periods of abelian functions of p variables is, if by ω we denote the $(p\times 2p)$ matrix of $2p$ strongly independent vector-periods, that there exist a rational skew-symmetric matrix R of order $2p$ for which $\omega R\omega'=0$, $i\bar{\omega}R\omega'>0$. The set $N\omega$ of such R satisfies $R_1, R_2\in N\omega\Rightarrow r_1R_2+r_2R_2\in N\omega$ for any rational $r_i>0$. Any set N of rational skew-symmetric matrices of order $2p$ satisfying such a condition is called a cone. If it is of the form $N\omega$ for some ω it is called admissible; the set of admissible cones was enumerated by A. A. Albert [Ann. of Math. (2) **35** (1934), 500–515]. For admissible N, let $\Omega_N=\{\omega|N\subseteq N\omega\}$, and let H_N be Ω_N reduced modulo equivalence $\sim$, where $\omega_1\sim\omega$ if $\omega_1=\alpha\omega$, α a non-degenerate square complex matrix. Let L_N be the group of integer-valued unimodular matrices U of order $2p$ such that $\omega\in\Omega_N\Rightarrow\omega U'\in\Omega_N$. Since the mapping $\omega\to\omega U'$ preserves equivalence, L_N may be regarded as operating on H_N. The quotient group of L_N by the normal subgroup of matrices inducing the identity map is called the modular group Γ_N, and the functions on H_N automorphic with respect to Γ_N are called modular functions.

The modular functions considered in this paper correspond to cones of the following description. Let k be a totally real algebraic number field of degree n, R_0 a rational skew-symmetric matrix of order $2p$, $p=np_0$, and $\alpha\to A(\alpha)$ a representation of k by rational matrices of order $2p$, where $AR_0=R_0A'$ for any matrix A of the representation. Then the set $N(k)$ of skew-symmetric matrices $R=AR_0$, where A runs over all matrices of the representation with positive eigenvalues, is an admissible cone, and $H_{N(k)}$ is the product $H_{p_0}{}^n$ of n generalized upper half planes of degree p_0. The modular group $\Gamma_{N(k)}$ is commensurable with the following special case of the above construction (two groups are commensurable if their intersection has a finite index in each). Consider the group of matrices U of order $2p_0$, with elements which are algebraic integers in k, satisfying

$$U'IU=I,\ U=\begin{pmatrix}A & B\\ C & D\end{pmatrix},\ I=\begin{pmatrix}0 & E\\ -E & 0\end{pmatrix}.$$

To every such U associate the transformation $(Z_1, \cdots, Z_n)\to(\tilde{Z}_1, \cdots, \tilde{Z}_n)$ of $H_{p_0}{}^n$ given by $\tilde{Z}_s=(A^{(s)}Z_s+B^{(s)})\times(C^{(s)}Z_s+D^{(s)})^{-1}$, where $V^{(s)}$ is the matrix whose elements are the sth conjugates of those of V. The group $\Gamma_{p_0}(k)$ of these transformations is here called a Hilbert-Siegel group, being due to Hilbert for $p_0=1$ and any k, to Siegel for any p_0 and $n=1$.

The main results, for any group Γ commensurable with $\Gamma_{p_0}(k)$, are: (1) For every $z_0\in H_{p_0}{}^n$ which is not a fixed point for any group transformation except the identity, there exist $m=np_0(p_0+1)/2$ functions regular at z_0, automorphic with respect to Γ, and with Jacobian $\neq 0$ at z_0. (2) If $z_1, z_2\in H_{p_0}{}^n$ are not equivalent under Γ, there exists an automorphic function h regular at z_1 and z_2 with $h(z_1)\neq h(z_2)$. (3) The field of functions automorphic with respect to Γ is isomorphic to a field of algebraic functions in m variables.

(3) was proved for $n=1$ by Siegel [Math. Ann. **116** (1939), 617–657; MR **1**, 203] and for $p_0=1$ by Maass [ibid. **117** (1940), 538–578; MR **2**, 87]; (1) and (2) were proved in a less precise form for $n=1$ by Siegel [op. cit.] and for $p_0=1$ by Steinitz [ibid. **71** (1912), 328–354]. The methods used in the present paper are developments of those of Siegel. To prove existence of modular functions, the author makes use of ordinary Poincaré series in a bounded domain, rather than (as did Siegel and Maass) of generalized Poincaré series in the upper half plane. Other simplifications are also achieved, in part by systematic use of majorant functions.

Adapted from the introduction

991b:

***Pyateckiĭ-Šapiro, I. I. Singular modular functions.** American Mathematical Society Translations, Ser. 2, Vol. 10, pp. 13–58. American Mathematical Society, Providence, R. I., 1958. iv+409 pp. $6.60.

Translation of #991 a.

992:

Röhrl, Helmut. Einige Bemerkungen über komplex-analytische Vektorraumbündel. Arch. Math. **8** (1957), 360–367.

Der Verf. bezeichnet mit X eine zusammenhängende algebraische Mannigfaltigkeit oder einen holomorph-vollständigen Raum und mit W ein q-rangiges komplex-analytisches Vektorraumbündel über X. $\mathfrak{S}_\omega(W)$ ist die Garbe der Keime lokaler holomorpher Schnitte in W, $\hat{\mathfrak{S}}_\omega(W)$ die Garbe der Keime lokaler meromorpher Schnitte über X. Es werde folgende Sätze bewiesen: (1) Der Modul $H^0(X, \hat{\mathfrak{S}}_\omega(W))$ über dem Körper $K(X)$ der in X meromorphen Funktionen hat den Rang q. (2) Jedes Element aus $H^1(X, \mathrm{GL}(q, C)_\omega)$ rührt von einem meromorphen "Divisor" her (Verallgemeinerung eines Resultates von K. Kodaira und D. C. Spencer). (3) Es sei $\hat{X}$, eine q-blättrige unbegrenzte analytische Überlagerung über X, deren Verzweigungspunkte nur über regulären Punkten von X und über einer positiv codimensionalen Untermannigfaltigkeit von X liegen. Dann ist der Körper $K(\hat{X})$ algebraisch vom Grade q über $K(X)$. Ist X vom Homologietypus der Zelle und holomorph-vollständig, so ist der Ring $I(\hat{X})$ der in $\hat{X}$ holomorphen Funktionen ein freier q-Modul über dem Ring $I(X)$ der in X holomorphen Funktionen. (4) Es sei W ein komplex-analytisches affines Faserbündel mit C^q als typischer Faser. Dann ist $H^0(X, \hat{\mathfrak{S}}_\omega(W))$ nicht leer. (5) Es sei X eine algebraische Mannigfaltigkeit, W ein konstantes $\mathrm{U}(q)$-Bündel über X. Dann ist $\dim_C H^0(X, \mathfrak{S}_\omega(W))=p$ genau, wenn $W\simeq W'\oplus(X\times C^p)$ und $W\not\simeq W''\oplus(X\times C^r)$, $r>p$ (mit $\simeq$ analytisch äquivalent; W', W'' komplex-analytische Vektorraumbündel; $\oplus$ Whitneysche Summe). Die Sätze gestatten Anwendungen auf Probleme der klassischen Funktionentheorie (Prymsche Funktionen, Existenz von Differentialformen, usw.). Ihre Vorraussetzungen lassen sich teilweise unter Benutzung einiger vor kurzem gewonnener Methoden abschwächen. *H. Grauert* (Princeton, N.J.)

993:

Martinelli, Enzo. Sulle varietà a struttura complessa. Ann. Mat. Pura Appl. (4) **43** (1957), 313–324.

Soit $\mathfrak{V}_n$ une variété analytique complexe de dimension complexe n et soit V_{2n} la variété réelle qu'elle définit. Considérons un vecteur tangent complexe $\mathfrak{a}$ en un point $\mathfrak{P}\in\mathfrak{V}_n$; soit a le vecteur tangent à V_{2n} image de $\mathfrak{a}$ au point P image de $\mathfrak{P}$; l'espace vectoriel réel de dimension 2

des vecteurs $u=\rho \exp i\theta\, a$ (ρ et θ réels) est appelé facette caractéristique (en abrégé f.c.) au point P. Deux vecteurs de la même f.c. sont dits de même longueur s'ils sont définis par le même nombre ρ, leur angle est la différence de leurs θ respectifs; la f.c. est orientée dans le sens des θ croissants; rapport de longueur, angle de deux vecteurs et orientation de la f.c. sont déterminés intrinsèquement par la structure complexe puisqu'ils ne dépendent pas du choix du vecteur a dans la f.c. Plus généralement, une structure presque complexe est déterminée par la donnée, en chaque point, d'un système de f.c. orientées. L'auteur établit une condition nécessaire et suffisante pour qu'une connexion affine analytique réelle sur V_{2n} conserve les f.c., et une aussi pour que, en outre, elle n'ait pas de torsion caractéristique (i.e., pas de torsion correspondant aux circuits appartenant à une f.c.); ces conditions sont des relations satisfaites par les coefficients de la connexion. Pour qu'une métrique sur V_{2n} détermine, sur chaque f.c., la notion d'angle définie ci-dessus par θ (métrique pseudo-conforme), il faut et il suffit qu'elle soit hermitienne. Une métrique hermitienne est kählérienne si et seulement si la connexion qu'elle définit conserve les f.c.

P. Dolbeault (Bordeaux)

994:

Martinelli, Enzo. Sulla curvatura delle superficie caratteristiche in una varietà kähleriana. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **21** (1956), 267–274.

Soit V une variété kählérienne; l'image réelle S_2 d'une courbe analytique complexe sur cette variété est appelée une surface caractéristique de V. En tout point P de S_2, considérons la f.c. tangente à S_2 [voir # 993 ci-dessus]. Considérons un arc de courbe Γ passant par P et un point $P' \in \Gamma$. Soit $\Delta\varphi$ l'angle, évalué par rapport à la métrique de l'espace euclidien tangent à V en P', que forme la f.c. tangente à S_2 en P' avec la facette obtenue par transport parallèle en P' de la f.c. tangente à S_2 en P, et soit Δs la longueur de l'arc PP', alors, l'expression $K_e=\lim_{P'\to P}(\Delta\varphi/\Delta s)$ qui ne dépend pas de la direction de la tangente en P à Γ est appelée courbure caractéristique de S_2 en P. La courbure caractéristique K_c, la courbure gaussienne K_f de S_2 et la courbure riemannienne K_e de V relativement à la facette tangente en P à S_2 sont liées par $K_f-K_e=-K_c^2$; en particulier, si V est un espace euclidien, la courbure gaussienne de S_2, égale à $-K_c^2$, est négative.

P. Dolbeault (Bordeaux)

995:

Matsushima, Yozô. Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kaehlérienne. Nagoya Math. J. **11** (1957), 145–150.

Soit V une variété à la structure complexe I. On désignera par $\mathfrak{a}$ l'ensemble de tous les champs de vecteurs (contrevariants) analytiques sur V.

Soit (V, g) une variété kaehlérienne, c'est-à-dire, le couple d'une variété complexe V et d'une métrique kaehlérienne g de V. On désignera par $\mathfrak{g}$ l'algèbre de Lie de tous les champs de vecteurs de Killing de la variété kaehlérienne (V, g). On dira qu'une variété kaehlérienne compacte satisfait à la condition (P) si l'on a $\mathfrak{a}=\mathfrak{g}+I\cdot\mathfrak{g}$.

Or, dans la présente Note l'auteur démontre trois théorèmes très intéressants suivants: (1) Soit (V, g) une variété kaehlérienne compacte dont la métrique g est d'Einstein. Alors (V, g) satisfait à la condition (P) et par suite le groupe d'homéomorphismes analytiques de V est réductif. (2) Soit (V, g) une variété kaehlérienne compacte et supposons que le premier nombre de Betti de V soit nul. Alors pour que (V, g) satisfasse à la condition (P), il faut et il suffit que $\mathfrak{a}^*=\mathfrak{a}^*\cap\delta C^2+\mathfrak{a}^*\cap dC^0$, où $\mathfrak{a}^*$ désigne l'espace vectoriel des 1-formes correspondant aux champs de vecteurs dans $\mathfrak{a}$, C^p ($p=0, 2$) désigne l'espace vectoriel des p-formes de V, et d et δ désignent les opérateurs de différentiation et de codifférentiation respectivement. (3) Soit V une variété complexe et compacte et supposons que le groupe $A(V)$ d'homéomorphismes analytiques de V soit transitif sur V. Si le groupe fondamental de V est fini et si la caractéristique d'Euler de V n'est pas nulle, le groupe de Lie $A(V)$ est semi-simple.

K. Yano (Paris)

996:

Lichnerowicz, André. Sur les transformations analytiques des variétés kählériennes compactes. C. R. Acad. Sci. Paris **244** (1957), 3011–3013.

The author gives a characterization of analytic infinitesimal transformations of a compact Kähler variety V_{2n} and proves the following theorems. (1) The identity component of all conformal transformations coincides with the identity component of the group of all automorphisms of the Kähler structure if $n>1$, and with the identity component of the group of analytic homeomorphisms in the case of a surface. (2) If V_{2n} has constant scalar curvature, the algebra of the analytic infinitesimal transformations is generated by the infinitesimal isometries and their images under J (the automorphism which is determined by the complex structure). This algebra is reductive. From this it follows as a corollary that under the hypothesis of (2), the dimension of the group of analytic transformations of V_{2n} is at most $2(n^2+2n)$. This maximum is attained in the case of the complex projective space.

This note generalizes results of Y. Matsushima [# 995 above].

W. M. Boothby (Evanston, Ill.)

SPECIAL FUNCTIONS

See also 1166, 1175, 1410, 1427, 1428, 1429.

997:

Ciorănescu, N. Sur un équation aux dérivées partielles du second ordre et une nouvelle généralisation des fonctions de Bessel. Bul. Inst. Politehn. Bucureşti **18** (1956), no. 1–2, 1–4. (Romanian. Russian and French summaries)

If the equation $(a\lambda r^2+1)\Delta u+4\lambda u=0$, where Δ is the Laplacian in polar coordinates and a and λ are constants, is solved by separating variables, then solutions of the radial part are

$$r^n\{1+\sum_1^\infty \frac{(-1)^k}{k!}\,\frac{[(n+1)a+1]\cdots[(k-1)(n+k-1)a+1]}{(n+k)!}\times(\lambda^{\frac{1}{2}}r)^{2k}\}.$$

These are the functions of the title.

N. D. Kazarinoff (Ann Arbor, Mich.)

998:

Vilenkin, N. J.; Akim, E. L.; and Levin, A. A. The matrix elements of irreducible unitary representations of the group of Euclidean three-dimensional space motions, and their properties. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 987–989. (Russian)

The author gives generalizations of some identities for Bessel functions which hold for functions appearing as matrix elements in the representations described in the title.

M. M. Day (Urbana, Ill.)

999:

Cohan, Norah V. The spherical harmonics with the symmetry of the icosahedral group. Proc. Cambridge Philos. Soc. **54** (1958), 28–38.

L'auteur applique la technique d'Altmann [même Proc. **53** (1957), 343–367; MR **19**, 133] aux développements en séries d'harmoniques sphériques pour toutes les représentations du groupe icosahédral. De nombreuses tables numériques ont été dressées. *R. Campbell* (Caen)

1000:

Sharma, J. L. Investigations in Lamé functions and the Whittaker's confluent hypergeometric functions. Bull. Allahabad Univ. Math. Assoc. **17** (1956/57), 1–3 (1958).

Summary of a thesis presented to the Allahabad University in 1938. The thesis contains 11 chapters, all of which have subsequently been published in European and Indian journals.

1001:

Arscott, F. M. Perturbation solutions of the ellipsoidal wave equation. Quart. J. Math. Oxford Ser. (2) **7** (1956), 161–174.

Ellipsoidal wave functions are doubly-periodic solutions of the differential equation

$$\frac{d^2w}{dz^2}=(a+bk^2\mathrm{sn}^2z+qk^4\,\mathrm{sn}^4\,z)w.$$

When $q=0$, these functions reduce to Lamé polynomials. The author expands both ellipsoidal wave functions and the corresponding characteristic values in powers of q, giving the first three terms of these expansions explicitly for the first eight functions. He describes the derivation of his formulas in sufficient detail to facilitate further development of the solutions. The paper also contains references to unpublished results of the author.

A. Erdélyi (Pasadena, Calif.)

1002:

Buchholz, H. Ein besonderes uneigentliches Integral über das Produkt zweier regulärer Coulombscher Wellenfunktionen. Z. Angew. Math. Mech. **38** (1958), 115–120. (English, French and Russian summaries)

The author evaluates the integral

$$\int_0^X M_{it,m}(-ix)M_{it,n}(-ix)x^{-k}dx$$

for $k=2$ in closed form and through a limiting process obtains the value of the infinite integral ($X=\infty$). He then evaluates the infinite integral ($X=\infty$) also for $k=3$. The latter integral arises in a problem of nuclear physics.

A. Erdélyi (Pasadena, Calif.)

ORDINARY DIFFERENTIAL EQUATIONS

See also 943, 990, 1154, 1296, 1363, 1416, 1417, 1445.

1003:

***Moulton, Forest Ray. Differential equations.** Dover Publications, Inc., New York, N.Y., 1958. xv+395 pp. $2.00.

A reprint of the well-known text published by Macmillan in 1930.

1004:

***Reuter, G. E. H. Elementary differential equations and operators.** Library of Mathematics. The Free Press, Glencoe, Ill; Routledge and Kegan Paul Ltd., London; 1958. viii+67 pp. $1.25; 5 s.

The main purpose of this book is to give a thoroughly elementary account of the 'operational' method for solving linear differential equations with constant coefficients... The book is addressed primarily to students of the exact sciences and of engineering, but it deals only with mathematical techniques and does not contain any illustrative examples drawn from other fields.

From the preface

1005:

***Lefschetz, Solomon. Differential equations: geometric theory.** Pure and Applied Mathematics. Vol. VI. Interscience Publishers, Inc., New York; Interscience Publishers Ltd., London; 1957. x+364 pp. $9.50.

If one could believe what the author says in the Preface, this book would merely be an enlarged edition of "Lectures on differential equations" [Princeton Univ. Press, 1946; MR **8**, 68]; actually it is an entirely new book with a much wider and deeper scope. The title may also be somewhat misleading since the material covers several profound analytical results, even if the general spirit of the book may be correctly labeled as "geometric". The author succeeds in the purpose of eliminating as much as possible tedious manipulations with formulas and bringing out clearly the intrinsic significance of concepts, methods and theorems. The consistent use of vector-matrix notation, which is not usual in textbooks on differential equations, fits this purpose very well; the reviewer also finds it commendable that the author does not dwell too much on real-function-theoretic technicalities in order to obtain maximum generality of the statements: any moderately mature reader will be able to see to what an extent the analyticity assumptions may be avoided. The book covers topics which are not frequently found in the Western literature; particularly noteworthy in this respect are Chapters VI, VII and VIII. It is essentially self-contained and may be read by anyone adequately acquainted with the calculus.

The contents of the book are the following. Chap. I: Preliminaries (metric spaces; vector-matrix algebra and analysis; analytical functions of several variables; differentiable manifolds). Chap. II: Existence theorems and general properties of the solutions (Picard's method; existence, uniqueness, continuity, differentiability and analyticity of the solutions). Chap. III: Linear systems (including a discussion of adjoint systems and systems with constant or periodic coefficients). Chap. IV: Stability (definitions of different types of stability: simple, asymptotic, uniform, equi-asymptotic, exponentially asymptotic, conditional, orbital; stability of mappings). Chaps. V–VI: The differential equation $\dot{x}=Px+q(x;t)$, P a constant matrix (classical theorems on asymptotic stability; Lyapunov's and Poincaré's expansion theorems; basic theorems of Lyapunov's "second method"; Persidskiĭ-Dyhman theorems on quasi-stability and the "reduction principle", with application to the investigation of stability in critical cases, especially in the case where one characteristic root of the linear approximation vanishes). Chap. VII: The differential equation $\dot{x}=P(t)x+q(x;t)$, P a variable matrix (Perron's transformation which reduces P to a triangular matrix; equivalence in the linear case between (i) exponential stability, (ii) Perron's conditions on the diagonal elements of the

triangular matrix, (iii) the existence of a Lyapunov function, (iv) the existence of a quadratic Lyapunov function; Perron's theorem on asymptotic stability in the first approximation; brief mention of Lyapunov numbers, regular systems and their application to stability). Chap. VIII: Periodic systems and their stability (extension of Lyapunov's expansion theorem; stability properties of periodic solutions of periodic or autonomous systems; Poincaré's theorems on the existence of periodic solutions of a perturbed autonomous or periodic system; existence and stability of periodic solutions of quasi-linear systems near a singular point of the linear approximation, also in the case where the latter has a pair of purely imaginary characteristic roots). Chaps. IX–X: Two-dimensional autonomous systems (a thorough discussion of critical points, elementary and general, proper and improper, as well as limit sets, limit cycles and limit path-polygons; structural stability or roughness). Chap. XI: Autonomous and periodic equations of the second order (non-dissipative systems; equations of Liénard and van der Pol; a survey of the Cartwright-Littlewood theorems; equations of the Levinson-Smith type; subharmonics in a special case). Chap. XII: Oscillations in systems of the second order and methods of approximation (methods of Poincaré, van der Pol, Minorski, Krylov-Bogolyubov; a sketch of the theory of Mathieu's equation; relaxation oscillations for Liénard's equation). The book ends with two excellent Appendices: (I) Jordan normal form of matrices with emphasis on the real case, logarithm of a matrix; (II) Topological index in the plane, Brouwer's fixed point theorem, index of a surface. There is a score of proposed problems, most of them being actual theorems not so easy to prove, and an extensive Bibliography with almost 100 items.

It would be unfair not to point out certain errors of the book. The reviewer does not wish to dwell on the numerous printing errors and misspellings, some of which are quite annoying, but there are certain important mistakes, among them the following: 1. The statement of theorem (7.1), p. 92, does not include the assumption that there are no purely imaginary characteristic roots, although this assumption is used in the proof (the mistake is also present in Bellman's book from which the theorem is taken; actually the theorem is also true without the assumption but this requires a special argument). 2. The definition of a normal system of solutions on p. 143 is not correct. 3. The proof that $\eta(2\pi)=\eta(0)D$ on p. 151 is not correct because $x(2\pi; u)$ does not belong to the hyperplane H. 4. The proof on p. 154 contains several mistakes, in particular, equation (8.2), which should obviously play a central role in the proof, is not used at all. 5. In the discussion of elementary critical points on p. 177–178, at least case (4.1) is not correctly dealt with (the proof breaks down, for instance, in the very simple example $\dot{x}=-x$, $\dot{y}=-2y+x^2$, which cannot be linearized analytically).

To sum up, the reviewer considers that the merits of the book by far outweigh its shortcomings and that it constitutes a very fine contribution to the literature on this subject. *J. L. Massera* (Montevideo)

1006:

Zubov, V. I. On a method of investigation of the stability of the trivial solution in doubtful cases. Prikl. Mat. Meh. **22** (1958), 46–49. (Russian)

Consider a system

(1) $$\dot{x}=g(x, y, t),\quad \dot{y}=f(x, y, t),$$

where x is an n-vector and y a k-vector. It is assumed that f, g are known and continuous in $\|x\|$, $\|y\|\leq H$, $t\geq 0$, and $f(x, 0, t)=0$, $g(0, 0, t)=0$. Basically one is to compare the stability (more or less) at the origins of the obvious partial systems with that of the full system (1). The stability of complete (1), ordinary or asymptotic, is defined as usual. Consider now the partial system

(2) $$\dot{y}=f(x(t), y, t),$$

where $x(t)$ is continuously differentiable and $\|x(t)\|\leq H$. Then (2) is 'strongly' stable or 'asymptotically' stable at the origin if, whatever $\|x\|<$ a certain $H_1<H$, the usual stability or asymptotic stability conditions hold. The function $W(x, y, t)$ is strongly x-definite negative if there exists a k-vector function $\varphi(x)$ such that every component $\varphi_s>0$ for $x\neq 0$, and such that $W(x, y(x), t)$ is definite negative for any choice of $y(x)$ continuous, and such that componentwise $|y_s(x)|\leq\varphi_s(x)$. Theorem 1: If the following conditions hold: (a) the origin is strongly stable [asymptotically stable] for (2); (b) there exists $V(x; t)$ positive definite, of class C^1, uniformly continuous in t for $x=0$, $\to 0$ with x uniformly in t for $t\geq 0$, and such that $\dot{V}=W(x, y, t)$ ($\dot{V}$ along the trajectories of (1)) is strongly x-definite negative; then the origin is stable [asymptotically stable] for (1). Theorem 2: Let $y(t, y^0, t_0)$ be the solution for (2) taking the value y^0 for $t=t_0$. If there is a positive ε such that we do not have $\|y(t, y^0, t_0)\|<\varepsilon$ for all $t\geq t_0\geq 0$ and $y^0\neq 0$, and this for all $\|x(t)\|<H_2$ small enough, then the origin is unstable for (1). Two other analogous stability and instability theorems are given in the paper. [References: Malkin, Prikl. Mat. Meh. **6** (1942), 411–448; MR **4**, 225; Zubov, Dokl. Akad. Nauk SSSR **100** (1955), 857–859; MR **16**, 924.]

{Reviewer's remark. The author does not seem to be aware of the related work of Persidskiĭ, Izv. Akad. Nauk Kazah. SSR **1951**, no. 62, Ser. Mat. Meh. **5**, 3–24 [MR **14** 753]. See in particular the reviewer's book (# 1005 above), p. 127, which contains analogous results to those of the paper.} *S. Lefschetz* (Mexico City)

1007:

Okiljević, Blažo. Contribution à la théorie de S. Lie sur les transformations infinitésimales pour l'intégration des équations différentielles ordinaires. Bull. Soc. Math. Phys. Serbie **6** (1954), 185–198. (Serbo-Croatian. French summary)

1008:

Zamfirescu, Ion. Un théorème de point fixe dans la théorie des équations différentielles et des équations aux dérivées partielles. Acad. R. P. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **9** (1957), 321–237. (Romanian. Russian and French summaries)

Il s'agit d'un théorème d'existence. Si les fonctions $f_i(x, y_i)$ sont "approximativement continues en projection" (notion définie par l'auteur), il existe une solution locale du système $dy_i/dx=f_i$, verifiant les données de Cauchy. Généralisation pour les systèmes d'équations aux dérivées partielles. *O. Bottema* (Delft)

1009:

Wintner, Aurel. On the expansion of solutions of ordinary differential equations according to powers of the initial constants or of parameters. Riv. Mat. Univ. Parma **7** (1956), 271–282.

Consider the differential equation $x'=f(x, \mu; t)$, where x and μ are complex, and let $x(t, \mu)$ denote the solution

satisfying the initial condition $x(0, \mu) \equiv 0$. (A change of notation allows μ to be interpreted as the initial condition on a solution of $x' = f(x; t)$.) It is assumed that f is continuous in $(x, \mu; t)$ on the set $|x|<1$, $|\mu|<1$, $t \in I=[0, a]$ $(a>0)$; satisfies $|f|<1$ there; is regular in (x, μ) on $|x|<1$, $|\mu|<1$, for each $t \in I$; and satisfies $f(0, 0; t)=0$ for each $t \in I$. A well-known result, due to Poincaré, is that for any $a>0$ there is an $\alpha>0$ such that $x(t, \mu)$ can be expanded uniquely in powers of μ, the expansion being valid for $t \in I$ and $|\mu|<\alpha$. The present paper is a rather thorough analysis of the size of α, since this number may be taken larger than what is indicated by Poincaré's proof. The basic tool used is the fact that if $g(z)$ is regular and $|g(z)|<C$ for $|z|<R$, then $|g'(z)|<CR/(R^2-|z|^2)$ for $|z|<R$. The author points out that his proofs remain essentially the same if x, f and μ are vectors.

C. E. Langenhop (Ames, Iowa)

1010:

Bandić, I. Über eine wichtige Differentialgleichung erster Ordnung. Bull. Soc. Math. Phys. Macédoine **7** (1956), 54–59. (Serbo-Croatian. German summary)

Il s'agit de l'équation diff. $y'^2+y^2=F^2(x)$, et l'auteur montre qu'elle s'intègre formellement par des quadratures si F satisfasse "une équation algébrique".

M. Tomić (Belgrade)

1011a:

Fage, M. K. Operator-analytical functions of one independent variable. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 1008–1011. (Russian)

1011b:

Fage, M. K. Integral representations of analytical operator functions of one variable. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 874–877. (Russian)

The author discusses linear differential operators defined on an open real interval. He defines first a class of L-analytic functions and states expansion theorems in terms of series of L-powers. He considers equivalence in a local sense of two such operators. In the second paper he also considers similar operators M whose coefficient functions are analytic in a domain of the complex plane, and studies transformations of M to L which can be represented by integrals. *M. M. Day* (Urbana, Ill.)

1012:

Olver, F. W. J. Uniform asymptotic expansions of solutions of linear second-order differential equations for large values of a parameter. Philos. Trans. Roy. Soc. London. Ser. A. **250** (1958), 479–517.

The concern is with the existence of solutions of any of

$$\frac{d^2w}{dz^2} = \{u^2 + f(u, \theta, z)\}w$$

$$\frac{d^2w}{dz^2} = \{u^2 z + f(u, \theta, z)\}w$$

$$\frac{d^2w}{dz^2} = \frac{1}{z}\frac{dw}{dz} + \{u^2 + \frac{\mu^2-1}{z^2} + f(u, \theta, \mu, z)\}w$$

which are asymptotically and uniformly represented by series of the form

$$\text{(W)} \quad P(z)\left\{1+\sum_1^\infty \frac{A_s(\theta, \mu, z)}{u^s}\right\} + \frac{P'(z)}{u^2}\sum_0^\infty \frac{B_s(\theta, \mu, z)}{u^s},$$

where u is a large positive parameter, $|\mu|$ is bounded, θ is a parameter as to which f has uniform properties, and z is restricted to suitable domains in which f is regular and bounded. Here $P(z)$ is either of two (or three for the second equation) suitable solutions of the companion equation in which $f=0$. Such equations arise when a similar equation, with u complex and with θ lacking, is transformed by $u=|u|e^{i\theta}$, $z=z_1e^{-i\theta}$, so that the work covers implicitly (but not explicitly) asymptotic expansions in which u is complex.

The precise theorems that are proved, which are too long to be quoted, are along the general lines of those previously proved by the author and others (references in the present paper), and the chief interest lies probably in the methods, viz: (i) The coefficients A_s, B_s are determined by substitution in the differential equation; their values at some chosen z_0 may be left arbitrary. (ii) Bounds for the A_s, B_s are established, depending on s but independent of (or sometimes with explicit dependence on) z. (iii) A function of u (which need not be analytic) is determined for which (W) is the asymptotic expansion at $z=z_0$, and another function is determined similarly related to the z-derivative of (W); these functions provide the initial conditions for a solution w of which (W) is to be proved the asymptotic expansion. (iv) A non-homogeneous differential equation is set up for the difference between w and an arbitrary truncation of (W), and by estimating this difference the asymptotic property is proved. {The proof of (ii) is slightly defective in that a sequence (η_s) is required instead of a single constant η.}

T. M. Cherry (Melbourne)

1013:

Donskaya, L. I. On the structure of the solution of a system of n linear differential equations in the neighborhood of a regular singular point. Vestnik Leningrad. Univ. Ser. Mat. Fiz. Him. **9** (1954), no. 8, 55–64. (Russian)

1014:

Skačkov, B. N. Qualitative illustration of the behavior of integral curves in the neighborhood of a singular point for one particular case. Vestnik Leningrad. Univ. Ser. Mat. Fiz. Him. **9** (1954), no. 8, 65–69. (Russian)

1015:

Andreev, A. F. Investigation of the behaviour of the integral curves of a system of two differential equations in the neighbourhood of a singular point. Amer. Math. Soc. Transl. (2) **8** (1958), 183–207.

Translated from Vestnik Leningrad. Univ. **10** (1955), no. 8, 43–65 [MR **17**, 364].

1016:

Gamkrelidze, R. V. On the theory of optimal processes in linear systems. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 9–11. (Russian)

The author considers the problem of choosing the control vector y so as to transform x, the solution of $\dot{x}=Ax+y$, from an initial state c into a final state c' in minimum time. This problem has been extensively discussed by a number of authors in papers not known or available to the author. The methods used by the author, and similar results, may be found in Bellman, Glicksberg and Gross, Quart. Appl. Math. **14** (1956), 11–18 [MR **17**, 1206] where further references will be found. These results in turn have been extensively generalized by Bass and Kalman in papers as yet unpublished.

R. E. Bellman (Santa Monica, Calif.)

1017:

Gel'man, A. E. The reducibility of a certain class of simultaneous differential equations containing quasi-periodic coefficients. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 535–537. (Russian)

The author discusses the question of reducibility in the sense of N. P. Erugin [Trudy Mat. Inst. Steklov **13** (1946); MR **9**, 509] of systems of linear homogeneous first order differential equations with almost periodic coefficients. For systems of order 2, say, $\dot{x}_i=P_{i1}(t)x_1+P_{i2}(t)x_2$, $i=1, 2$, P_{ij} almost periodic, the question of reducibility is connected with the existence of almost periodic solutions of the Riccati equation $\dot{\tau}=P_{12}+(P_{22}-P_{11})\tau-P_{21}\tau^2$ and with the arithmetic nature of the characteristic exponents in the Fourier series of the coefficients P_{ij}. Criteria and examples illustrate the results. No proofs are given.

L. Cesari (Lafayette, Ind.)

1018:

Kreĭn, M. G. On the characteristic function $A(\lambda)$ of a linear canonical system of differential equations of second order with periodic coefficients. Prikl. Mat. Meh. **21** (1957), 320–329. (Russian)

Consider equations (1) $\dot{x}=J(H_0(t)+\lambda H_1(t))x$, (2) $\ddot{y}+\lambda p(t)y=0$, (3) $\ddot{y}-r(t)y+\lambda p(t)y=0$, where x is a two-dimensional vector, y a scalar, $J=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, λ a real parameter; H_0, H_1 are symmetric matrices and p, r scalars, all of them periodic of period T and absolutely integrable on $[0, T]$; let $A(\lambda)=\frac{1}{2}(\rho_1+\rho_2)$, where ρ_1, ρ_2 are the multipliers of the equations; the condition for stability is then $|A(\lambda)|<1$. The following property (c) of A is investigated: If λ is a stationary value of A, then $|A(\lambda)|\geqq 1$, $A(\lambda)A''(\lambda)<0$. Equations (1), (2), (3) have property (c), respectively, if (i) H_1 is positive semi-definite and the equation $\dot{x}=JH_0x$ has no non-trivial solution for which $H_1x=0$ almost everywhere; (ii) $0<\int_0^T|p(t)|dt<\infty$; (iii) either $p\geqq 0$ and $\int_0^T p(t)dt>0$, or $\int_0^T(y'^2+ry^2)dt>0$ for any continuously differentiable y which is not identically zero.

J. L. Massera (Montevideo)

1019:

Krohová-Šantavá, Sylva. Über die Wurzeleigenschaften der Lösungen eines Systems von zwei linearen Differentialgleichungen erster Ordnung. Publ. Fac. Sci. Univ. Masaryk **1955**, 429–449. (Russian. German summary)

1020:

Laitoch, Miroslav. Die Identität der Zentraldispersionen erster und zweiter Art, die zu der Differentialgleichung zweiter Ordnung $y''=Q(x)y$ gehören. Czechoslovak Math. J. **6(81)** (1956), 365–380. (Russian. German summary)

1021:

Laitoch, Miroslav. Über gewisse Lösungen der Funktionalgleichung $F[\varphi(x)]-F(x)=1$. Časopis Pěst. Mat. **81** (1956), 420–425. (Czech. Russian and German summaries)

In der vorliegenden Arbeit werden zwei Sätze über den Zusammenhang der Abelschen Funktionalgleichung (1) $F(\varphi(x))=F(x)+1$ mit der Theorie der linearen Differentialgleichung 2. Ordnung (2) $y''=q(x)y$, q stetig, abgeleitet. Im wesentlichen geht es um das folgende Resultat: Sind die Lösungen von (2) oszillatorisch und ist F eine wachsende Lösung der nichtlinearen Differentialgleichung 3. Ordnung $-\{F, x\}-\pi^2F'^2=q(x)$, so stellt die zu der Differentialgleichung (2) gehörige erste Zentraldispersion 1. Art, $\varphi(x)$, die einzige Lösung der Funktionalgleichung (1). (Die erste Zentraldispersion 1. Art, φ, ist so definiert: $\varphi(x)$ ist die erste nach x folgende Nullstelle eines jeden in x verschwindenden Integrals von (2). [Vgl. Borůvka, Czechoslovak Math. J. **3(78)** (1953), 199–255; MR **15**, 706].)

O. Borůvka (Brno)

1022:

Kurzweil, Jaroslav. Sur l'équation $\ddot{x}+f(t)x=0$. Časopis Pěst. Mat. **82** (1957), 218–226. (Czech and Russian summaries)

Der Verfasser beschäftigt sich mit der Frage, unter welchen Bedingungen alle Lösungen von $\ddot{x}+f(t)x=0$ bei $t\to\infty$ gegen Null konvergieren. Er modifiziert einen Beweis von Tonelli [Scritti matematici offerti a Luigi Berzolari, Ist. Mat. Univ. Pavia, 1936, pp. 404–405] um zu beweisen, dass $f\in F$ eine dazu hinreichende Bedingung ist. Dabei gehört eine stetige nicht abnehmende Funktion f zu der Klasse F, wenn man $\varepsilon>0$ so auswählen kann, dass $\int_H d\log f=\infty$ für jede offene Menge H mit $\liminf_{t\to\infty}\mu(H\cap\langle t, t+1\rangle)>1-\varepsilon$ gilt.

M. Zlámal (Brno)

1023:

Ráb, Miloš. Eine Bemerkung zu der Frage über die oszillatorischen Eigenschaften der Lösungen der Differentialgleichung $y''+A(x)y=0$. Časopis Pěst. Mat. **82** (1957), 342–348. (Czech. Russian and German summaries)

Der Verfasser verallgemeinert zwei Sätze, welche die Oszillation der Integrale von $y''+A(x)y=0$ betreffen. Der erste ist ein Kriterium des Rezensenten [Časopis Pěst. Mat. Fys. **75** (1950), 213–218; MR **13**, 132], den zweiten bewies Nikolenko [Ukrain Mat. Ž. **7** (1955), 124–127; MR **17**, 263].

M. Zlámal (Brno)

1024:

Horváth, Ján. On inverse problems of the Sturm type. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **6**(1956), 208–254. (Slovak)

This is a systematic exposition of known results on the inverse Sturm-Liouville problem, with a discussion of the physical applications. Some results have been made more precise and the method of presentation is in places new.

František Wolf (Berkeley, Calif.)

1025:

Faddeev, L. D. An expression for the trace of the difference between two singular differential operators of the Sturm-Liouville type. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 878–881. (Russian)

Generalizing some results of Gel'fand and Levitan [same Dokl. (N.S.) **88** (1953), 593–596; MR **15**, 33] to some differential operators on the positive reals, the author defines a trace for some operators of the form $Ly=-y''+q(x)y$, $y(0)=0$ which have continuous spectrum.

M. M. Day (Urbana, Ill.)

1026:

Putnam, C. R. Continuous spectra and unitary equivalence. Pacific J. Math. **7** (1957), 993–995.

Let $p(t)>0$ and $f(t)$ be real-valued functions on $0\leqq t<\infty$, the latter being continuous, and let λ be a real parameter. Suppose that the differential equation

$$(1)\qquad (px')'+(\lambda+f)x=0$$

is of the limit point type at $t=+\infty$, and consider the boundary value problem B_α given by (1) and the boundary condition $x(0)\cos\alpha+x'(0)\sin\alpha=0$, where $0\leqq\alpha<\pi$. Let S_α be the spectrum, and H_α the self-adjoint operator

belonging to B_α. Suppose there is a fixed couple α, β $(0 \leq \alpha < \beta < \pi)$ such that H_α and H_β have a purely continuous spectrum, which by a result of H. Weyl implies that their spectra are identical. The object of the paper is to show that then H_α and H_β are unitarily equivalent provided that the following two additional conditions are satisfied: (i) neither spectrum covers the whole infinite line, and (ii) the monotone non-decreasing function of λ which determines the continuous spectrum [see, e.g., Titchmarsh, Eigenfunction expansions associated with second order differential equations, Oxford, 1946, p. 58; MR 8, 458] is absolutely continuous for H_α and H_β. The proof consists in showing that the difference $H_\beta^{-1} - H_\alpha^{-1}$ satisfies sufficient conditions for unitary equivalence given by Rosenblum [Pacific Math. J. 7 (1957), 997–1010; MR 19, 756]. Finally, the author illustrates his result by discussing the case where $f(t) \equiv 0$.

E. H. Rothe (Ann Arbor, Mich.)

1027:

Ešukov, L. N. On the existence of a solution of a boundary-value problem for systems of ordinary linear differential equations. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 313–319. (Russian)

Es handelt sich um die Randwertaufgabe (1) $dx_i/dt = \sum_{k=1}^{n} p_{ik}(t)x_k$, (2) $x_1(t_i) = A_i$ $(i = 1, \cdots, n)$ und es wird das folgende Problem gelöst: Unter welchen Bedingungen existiert die Lösung von (1), (2) bei allen $t_1, \cdots, t_n$ aus gewissem Intervall $\langle a, b \rangle$ und bei beliebigen $A_1, \cdots, A_n$? Eine Antwort lautet: Wenn jede Hauptminor der Matrix $\|x_{1k}^{(i-1)}\|$ in $\langle a, b \rangle$ von Null verschieden ist. Dabei bedeuten $x_{11}, \cdots, x_{1n}$ die erste Zeile der Fundamentalmatrix von (1). Weitere hinreichende Bedingungen sind für den Fall von konstanten Koeffizienten angegeben.

M. Zlámal (Brno)

1028:

Laasonen, Pentti. Bemerkung zur iterativen Lösung der Eigenwertaufgabe einer Vektordifferentialgleichung. Ann. Acad. Sci. Fenn. Ser. A. I. no. **230** (1956), 8 pp. (1957).

In some previous papers [same Ann. nos. **133** (1952), **195** (1955); MR **15**, 33; **17**, 489] the author has considered the iterative solution of the eigenvalue problem connected with a homogeneous system of linear first-order differential equations with homogeneous boundary conditions, and in the present paper the problem of the convergence of the iterative solutions towards the associated eigenfunctions is discussed in greater detail. *P.-O. Löwdin* (Uppsala)

1029a:

Pontryagin, L. S. Asymptotic behavior of solutions of systems of differential equations with a small parameter in the derivatives of highest order. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 605–626. (Russian)

1029b:

Miščenko, E. F. Asymptotic calculation of periodic solutions of systems of differential equations containing small parameters in the derivatives. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 627–654. (Russian)

The van der Pol equation for extreme value of the parameter is equivalent to the system $\varepsilon\dot{x} = y - (\frac{1}{3}x^3 - x)$, $\dot{y} = -\varepsilon x$, where ε is psitive and very small. The position of the (unique) limit cycle for $\varepsilon \to 0$, and the period of the oscillation were asymptotically obtained by Jules Haag [Ann. Sci. Ecole Norm. Sup (3) **60** (1943), 35–111; MR **7**, 299], an important contribution unknown to the authors, and later also by Dorodnytzin [Akad. Nauk SSSR. Prikl. Mat. Meh. **11** (1947), 313–328; MR **9**, 144], whom they quote. The present papers both deal with a far-reaching and novel generalization of the work of Haag and Dorodnytzin. They consider the same system

$$\varepsilon\dot{x} = f(x, y), \tag{1a}$$

$$\dot{y} = g(x, y), \tag{1b}$$

where ε is small and x, y are a k- and l-vector. The first paper deals with the following problem. Let $x = x(t, \varepsilon)$, $y = y(t, \varepsilon)$ be a solution of (1). Does it tend to a certain limit as $\varepsilon \to 0$? Observe that the variables x, y do not behave alike, for one may write the velocity vector as

$$v = (\varepsilon^{-1} f(x, y), 0) + (0, g(x, y))$$

so that by and large the x-component of v is very rapid, while its y-component is slow (independent of ε). One approach to the question is then to consider first the behavior of the rapid variable x when the slowly varying one y is kept constant. Thus one takes up first (1a) alone. The explicit assumption is made that the stationary solutions of (1a) are only positions of stable equilibrium to which all solutions tend as $t \to +\infty$. Let $x = \phi(y)$ be a stable position of equilibrium of (1a). It depends on y and as y varies in a certain region Γ this property persists. Substituting in (1b) there follows $\dot{y} = g(\phi(y), y)$. Let it have the solution

$$y = \psi(t). \tag{2}$$

Inserting in ϕ there follows

$$x = \phi(\psi(t)), \tag{3}$$

and (2), (3) together constitute an approximate solution of (1) to order ε [proof by Tihonov, Mat. Sbornik N.S. **31(73)** (1952), 575–586; MR **14**, 1085; see also Vasil'eva, ibid., 587–644; MR **14**, 1086]. The approximate solution thus obtained is an exact solution of the degenerate system

$$f(x, y) = 0, \quad \dot{y} = g(x, y). \tag{4}$$

The operation just described is valid as long as the characteristic roots of the critical point $x = \phi(y)$ of (1a) have negative real parts. However, if on $t \in [-\alpha, 0]$ as $t \to 0$ one of these roots $\to 0$, the scheme breaks down. In the present paper the assumption is made that exactly one of the roots tends then to zero. Let $x = x_0(t)$, $y = y_0(t)$ be a solution of the degenerate system along which this happens. Let the initial point for a solution of (1) differ to order ε from that of the degenerate solution, and let it be

$$x = x(t, \varepsilon), \quad y = y(t, \varepsilon).$$

The proof is given in the paper that there is an asymptotic representation valid even for $t = 0$ with small terms of order $\varepsilon^{2/3}$ and $\varepsilon \log \varepsilon$, to order ε.

The second paper deals with $x_0(t)$, $y_0(t)$: A trajectory of the kind considered limiting in the first paper consists of alternating arcs described instantaneously and slowly. Such trajectories may well be closed. If Z_0 is one, it is assumed that: (a) Z_0 is stable, in the sense that trajectories of (4) very near Z_0 remain near Z_0; more precisely, a transverse surface to Z_0 at a point P along the slow motion part undergoes (in the well known sense) a local topological "compressing" transformation ϕ around the (fixed) point on Z_0 and this is not changed when ϕ is linearized; (b) at P the characteristic roots of the matrix $\partial f/\partial x$ have negative real parts; (c) at the exceptional points (like points $t = 0$ of the preceding paper) the same matrix has exactly one root zero, the others behaving as before.

Under these conditions it is shown that for ε small there is a cycle Z_ε of (1) very near Z_0 and which $\to Z_0$ as $\varepsilon\to 0$. The solution corresponding to Z_ε and its period are then computed to within terms of order ε (terms of orders $\varepsilon^{2/3}$ and $\varepsilon \log \varepsilon$ are obtained).

[Additional references: Železcov and Rodygin, Dokl. Akad. Nauk SSSR (N.S.) **81** (1951), 391–392; Miščenko and Pontrjagin, ibid. **102** (1955), 889–891; MR **17**, 153.]

S. Lefschetz (Princeton, N.J.)

1030:

Mitropol'skiĭ, Yu. A. L'action des forces extérieures à fréquences variables sur les systèmes oscillatoires non-lineaires. Bul. Inst. Politehn. Iaşi (N.S.) **3** (1957), no. 1–2, 15–24. (Russian. French and Romanian summaries)

This is essentially a lecture on the topic developed at length in the author's book, "Nonstationary processes in nonlinear oscillatory systems" [Izdat. Akad. Nauk Ukraïn. SSR, Kiev, 1955; MR **17**, 735]. The basic methods are the well-known methods of Krylov-Bogoliubov.

S. Lefschetz (Princeton, N.J.)

1031:

Bihari, I. Researches of the boundedness and stability of the solutions of non-linear differential equations. Acta Math. Acad. Sci. Hungar. **8** (1957), 261–278.

The author proves various results concerning the boundedness of all solutions and the stability of the trivial solution of equations of the type (*) $\dot{x}=A(t)x+B(t)\varphi(x)$, where $A(t)$, $B(t)$ are continuous for $t\geqq 0$ and $\varphi(x)$ is continuous for small $\|x\|$. Among them is the following theorem. If A is constant and all solutions of $\dot{y}=Ay$ are bounded, if $\int_0^\infty \|B(t)\|dt<k/\int_0^\infty dt/\omega(t)$, where $\omega(t)\geqq\max\{\|\varphi(x)\|:\|x\|\leqq t\}$, then all solutions $x(t)$ of (*) are bounded on $t\geqq 0$ provided $\|x(0)\|$ is sufficiently small. If $\varphi(0)=0$ and $\int_0^1 dt/\omega(t)$ diverges, $x(t)\equiv 0$ is stable. All proofs make use of a previous lemma of the author [same Acta **7** (1956), 81–94; MR **18**, 38].

H. A. Antosiewicz (Los Angeles, Calif.)

1032:

Corduneanu, C. Systèmes d'équations différentielles dont toutes les solutions sont bornées. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **9** (1957), 315–320. (Romanian. Russian and French summaries)

The author considers systems $\dot{x}=X(t,x)+f(t,x)$ where X and f are continuous for $t\geqq 0$ and all x. In the general case and when $X(t,x)$ is linear in x he proves that all solutions are bounded as $t\to\infty$ provided X and f satisfy certain conditions which imply that all solutions of $\dot{y}=X(t,y)$ are bounded as $t\to\infty$ and that f remains small in some sense for small enough $\|x\|$ and all $t\geqq 0$. This extends results of Vinograd [Uspehi Mat. Nauk (N.S.) **8** no. 1(53) (1953), 115–120; MR **16**, 132], Szmydt [Ann. Polon. Math. **2** (1955), 234–236; MR **17**, 616], Conti [Riv. Mat. Univ. Parma **6** (1955), 3–35; MR **17**, 969], and the reviewer [J. London Math. Soc. **31** (1956), 208–212; MR **18**, 42].

H. A. Antosiewicz (Los Angeles, Calif.)

1033:

Schäffer, Juan Jorge. On non-linear almost-periodic differential equations. Bol. Fac. Ingen. Agrimens. Montevideo **6**=Fac. Ingen. Agrimens. Montevideo. Publ. Inst. Mat. Estadist. **3** (1957), 17–52. (Spanish. English summary)

Viene considerato il sistema differenziale ordinario:

(1) $$\dot{x}+g[f(x),p(t)]=0,$$

con x, $f(x)$, $p(t)$, $g(u,v)$ vettori.

Sotto opportune ipotesi per $f(x)$, $g(u,v)$ e supposto $p(t)$ misurabile e limitato viene studiato il comportamento asintotico degli integrali del sistema (1).

Successivamente, supposta $p(t)$ quasi-periodica, in adeguate ipotesi per il codominio di $p(t)$, si dimostra che ogni integrale di (1) è asintotico, in un senso ben precisato, all'unico integrale quasi-periodico della (1). Questo risultato è caso particolare di un teorema più generale relativo ad un vettore $p(t)$ "asintoticamente quasi-periodico".

Vengono fatte applicazioni al sistema:

$$M\ddot{x}+N\dot{x}+h(x)+p(t)=0.$$

G. Fichera (Roma)

1034:

Konoplev, V. P. Asymptotic representation of solutions of second order linear differential equations. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 25–28. (Russian)

A discussion of the method of R. E. Langer for asymptotic solution of differential equations with turning points is given. In 1935 Langer [Trans. Amer. Math. Soc. **37** (1935), 377–416] discussed solutions of equations of the type (*) $Z''+[\lambda^2 z^{\beta} r(z)+sz^{-2}+q(z)z^{-1}]Z=0$, $\beta>-2$. A slightly modified version of his approximating equation is derived by the author, from which the behavior of the solution of (*) regular at $z=0$ may be more simply determined.

N. D. Kazarinoff (Ann Arbor, Mich.)

1035:

Voĭlokov, M. I. Sufficient conditions for the existence of exactly n limit cycles for the system $dx/dt=y$, $dy/dt=F(y)-x$. Mat. Sb. N.S. **44(86)** (1958), 235–244. (Russian)

Take the system

(1) $$\dot{x}=F(x)-y, \quad \dot{y}=x$$

equivalent to

$$\ddot{x}+f(x)\dot{x}+x=0, \quad F=\int_0^x f dx.$$

Under continuity and other special assumptions the author determines $n+1$ mutually inclusive Jordan curves $J_0, \cdots, J_n$ such that: (a) J_0 is a small circle surrounding the origin (the only critical point); (b) if S_i is the region between J_{i-1} and J_i, $i\geqq 1$, then the S_i are alternatingly entered or left by the trajectories across J_{i-1} and J_i; (c) each S_i contains one limit-cycle (Poincaré) and only one. [References: Levinson and Smith, Duke Math. J. **9** (1942), 382–403; MR **4**, 42; Ryškov, Dokl. Akad. Nauk SSSR **96** (1954), 921–924; MR **16**, 362; Markosyan, Certain questions in the qualitative investigation of the system of two nonlinear differential equations (1955); Russian thesis); Massera, Boll. Un. Mat. Ital. (3) **9** (1954), 367–369; MR **16**, 925.]

S. Lefschetz (Princeton, N.J.)

1036:

***Petrovskiĭ, I. G.; and Landis, E. M. On the number of limit cycles of the equation $dy/dx=P(x,y)/Q(x,y)$, where P and Q are polynomials of the second degree.** American Mathematical Society Translations, Ser. 2, Vol. 10, pp. 177–221. American Mathematical Society, Providence, R.I., 1958. iv+409 pp. $6.60.

Translated from Mat. Sb. N.S. **37(79)** (1955), 209–250; MR **17**, 364.

1037:

Aĭzerman, M. A.; and Gantmaher, F. R. Determination of periodic solutions in systems with a straight-line characteristic composed of line pieces parallel to two given straight lines. II. Avtomat. i Telemeh. **18** (1957), 193–200. (Russian. English summary)

In a previous paper [Avtomat. i Telemeh. **18** (1957),

97–110] the authors described a method for finding the periodic solutions of systems with a non-linear characteristic f of type:

$$\dot{x}=Ax+\lambda f(x_1)+F(t)$$

where x, λ, F are n-vectors, A a constant matrix, F continuous and smooth. The method was based upon the use of a complete Fourier series. This method is now extended to a variety of broken line characteristics whose sides are parallel to one of two given directions.

S. Lefschetz (Mexico City)

1038:

Zubov, V. I. A method of integration of differential equations describing the oscillations of mechanical systems with distributed parameter. Vestnik Leningrad. Univ. Ser. Mat. Fiz. Him. **9** (1954), no. 2, 69–75. (Russian)

1039:

Faure, Robert. Sur les systèmes d'équations différentielles non linéaires à coefficients périodiques. Étude d'un cas particulier. C. R. Acad. Sci. Paris **245** (1957), 1588–1590.

The author considers the system

$$(1)\qquad \frac{dx_i}{dt}+\lambda\sum_k p_{ik}x_k=\lambda f_i(x_k, t)\quad (i, k=1, \cdots, n),$$

where λ is a real parameter and $f_i(x_k, t)$ are periodic functions with the period T. He proves that there exists a periodic solution of the same period if $|\lambda|$ is sufficiently small or if the frequency is sufficiently large.

M. Zlámal (Brno)

1040:

Ginzburg, I. P. On sufficient conditions for stability for the trivial solutions of a linear homogeneous differential equation of nth order and systems of n-homogeneous differential equations with variable coefficients. Vestnik Leningrad. Univ. Ser. Mat. Fiz. Him. **9** (1954), no. 5, 53–65. (Russian)

1041:

Pliss, V. A. Relation between the region of stability and the region of uniform analyticity with respect to the initial data for solutions of systems of differential equations. Vestnik Leningrad. Univ. Ser. Mat. Fiz. Him. **9** (1954), no. 11, 19–32. (Russian)

1042:

Corduneanu, C. Sur la notion de stabilité. Rev. Math. Pures Appl. **2** (1957), 497–500.

In the present paper the author proposes a concept of stability which is a weaker version of the strong stability of Malkin and Massera. The concept is obtained from the mentioned one by considering only solutions (both in the perturbed as well as in the unperturbed system) which satisfy a given property P. A solution $x(t)$, $x=(x_1, \cdots, x_n)$, $0\leqq t<+\infty$, of system $dx/dt=f(t, x)$ is said to be conditionally stable with respect to property P provided $x(t)$ has property P (for instance, boundedness and given $\varepsilon>0$ there is $\delta(\varepsilon)>0$ as follows: for $!\,R(t, y)$ of some class (say continuous) with $\|R(t, y)\|<\delta(\varepsilon)$ for all $t\geqq 0$, $\|y-x(t)\|<\varepsilon$; and for all solutions $y(t)$ of $dy/dt=f(t, y)+R(t, y)$, with $\|y(0)-x(0)\|<\delta(\varepsilon)$, which exist in $[0, +\infty)$ and which have property P (if any); we have $\|y(t)-x(t)\|<\varepsilon$ for all $t\geqq 0$. Examples are given.

L. Cesari (Lafayette, Ind.)

1043:

Wintner, Aurel. On Gusarov's stability theorem. Bul. Inst. Politehn. Iaşi (N.S.) **3** (1957), no. 1–2, 25–30. (Russian and Romanian summaries)

The author points out that a result due to Gusarov, as well as an extension due to Zlámal, can be obtained by means of results he obtained in Phys. Rev. **72** (1947), 516–517 [MR **9**, 143]. Subsequent proofs by Bellman and Burati, also based on the use of the Liouville transformation, are particular applications of the results in the paper cited above. *R. Bellman* (Santa Monica, Calif.)

1044:

Vejvoda, Otto. The stability of solutions of a system of differential equations in the complex domain. Časopis Pěst. Mat. **82** (1957), 137–159. (Czech. Russian and English summaries)

Es wird die Stabilität der trivialen Lösung des Systems

$$\frac{dz_j}{dt}=\sum_{k=1}^{n} c_{jk}z_k+Z_j(t, z_1, \cdots, z_n)\ (j=1, \cdots, n)$$

untersucht. Dabei sind z_j komplexe Funktionen der reellen Veränderlichen t, c_{jk} komplexe Konstanten, und Z_j komplexe Funktionen, die gewisse Bedingungen erfüllen. Mit dieser Frage hat sich schon Perron [Math. Z. **29** (1928), 129–160] beschäftigt. Der Verfasser wendet die zweite Liapounoffsche Methode an um erstens von neuem die Sätze aus der Theorie der ersten Annäherung zu beweisen, zweitens kritische Fälle, und zwar den Fall einer Nullwurzel und einer rein imaginären Wurzel, zu untersuchen. *M. Zlámal* (Brno)

1045:

***Yakubovič, V. A. Questions of the stability of solutions of a system of two linear differential equations of canonical form with periodic coefficients.** American Mathematical Society Translations, Ser. 2, Vol. 10, pp. 125–175. American Mathematical Society, Providence, R. I., 1958. iv+409 pp. $6.60.

Translated from Mat. Sb. N.S. **37(79)** (1955), 21–68; MR **17**, 483.

1046:

Lykova, O. B. On the behaviour of solutions of a system of differential equations in the neighbourhood of isolated static solution. Dokl. Akad. Nauk SSSR (N.S) **115** (1957), 447–449. (Russian)

Die Verfasserin betrachtet Systeme von Differentialgleichungen vom Typus

$$(*)\qquad \frac{dx}{dt}=X(x)+\varepsilon X^*(t, x, \varepsilon)\ (\varepsilon \text{ klein und positiv}).$$

Von den "ungestörten" Gleichungen $dx/dt=X(x)$ wird die Existenz einer isolierten stabilen Lösung vorausgesetzt, wobei mit $x=0$ die Beziehungen $X(0)=0$, $X_x'(0)\neq 0$ gelten. Die Funktionen $X(x)+\varepsilon X^*(t, \varepsilon, x)$ sind für $-\infty<t<\infty$ und für x innerhalb $U\sigma_0$ erklärt, wobei $U\sigma_0$ eine σ_0-Umgebung des Punktes $x=0$ darstellt. In t wird Periodizität vorausgesetzt mit der Periode 2π, in x, ε stetige Differenzierbarkeit bis zu beliebiger Ordnung. Analoge Voraussetzungen bestehen für Gleichungen in δx. Unter diesen Annahmen kann das System (*) in einem gewissen Bereich $\mathfrak{R}_t$ durch das äquivalente System

$$\frac{d\xi_t}{dt}=i\omega\xi_t+P_1(t, \xi_t, \xi_t^*, \varepsilon)$$

$$\frac{d\xi_t^*}{dt} = -i\omega\xi_t^* + Q_1(t, \xi_t, \xi_t^*, \varepsilon)$$

ersetzt werden ($x = A\xi + B\xi^* + D\bar{h}$, $\delta x = C_1 A e^{i\omega t} + C_2 B e^{-i\omega t} + D\bar{h}$). Für $\mathfrak{M}_t$ wird eine Parameterdarstellung angegeben. Lösungen, die hinreichend nahe an $\mathfrak{M}_t$ definiert sind, werden mit solchen aus $\mathfrak{M}_t$ verglichen. Von besonderem Interesse ist der Fall $X(x) = P \cdot x$, wobei P eine konstante Matrix $\|p_{jk}\|$ bedeutet. *M. Pinl* (Köln)

1047:

Kostomarov, D. P. On boundary problems for eigenvalues and characteristic functions of ordinary differential equations containing a small parameter in the highest derivative term. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 230–233. (Russian)

The connection between the two following eigenvalue problems is studied here: (I) $\mu L[u] + l[u] = \lambda\rho(x)u$, $u^{(s)}(a) = u^{(s)}(b) = 0$ $(0 \le s \le n-1)$; (II) $l[v] = \lambda\rho(x)v$, $v^{(s)}(a) = v^{(s)}(b) = 0$ $(0 \le s \le m-1)$; where $L[u] = (-1)^n \sum_{i=0}^{2n} p_i(x) u^{(i)}(x)$, $l[u] = (-1)^m \sum_{i=0}^{2m} q_i(x) u^{(i)}(x)$, $m > 0$, $n - m = k > 0$, and μ is a small positive parameter. Such a problem with more general boundary conditions was already investigated by Moser [Comm. Pure Appl. Math. **8** (1955), 251–278; MR **17**, 154]. — Let $\lambda_j(\mu)$ and λ_j denote the eigenvalues of the problem (I) and (II), respectively, and $v_j(x, \lambda)$ be a fundamental set of solutions of the equation $l[v] = \lambda\rho(x)v$. The author introduces without a proof the following theorem: Let the equation $\det \left\| v_i^{(s)}(a, \lambda) \right\| = 0$ $\left\| v_i^{(s)}(b, \lambda) \right\|$ $(1 \le i \le 2m,$ $0 \le s \le m-1)$ have only simple roots in an open bounded region G. Then μ being sufficiently small $(0 < \mu < \mu_0(G))$ both problems (I) and (II) have the same number of eigenvalues in G, and $\lambda_j(\mu) = \lambda_j + \mu^{1/2k}\lambda_{j1}(\mu)$ where $\lambda_{j1}(\mu)$ are analytic and bounded functions. This theorem establishes a one-to-one correspondence between the eigenvalues of problem (I) and (II), respectively. More precise formulae are introduced in the case of the self-adjoint operator $l[u]$. *M. Zlámal* (Brno)

1048a:

Volosov, V. M. On the theory of nonlinear differential equations of higher orders with a small parameter in the highest derivative. Amer. Math. Soc. Transl. (2) **8** (1958), 209–241.

1048b:

Volosov, V. M. On solutions of some differential equations of the second order depending upon a parameter. Amer. Math. Soc. Transl. (2) **8** (1958), 243–255.

These two papers are translated from Mat. Sb. N.S. **31** (73) (1952), 645–674, 675–686 [MR **14**, 1086].

1049:

Mitrinović, Dragoslav S. Sur certaines équations différentielles indéterminées. Bull. Soc. Math. Phys. Serbie **7** (1955), 171–178 (1956). (Serbo-Croatian. French summary)

1050:

Sobolevskiĭ, P. E. Approximate methods of solving differential equations in Banach space. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 240–243. (Russian)

Verfasser untersucht die Differentialgleichung

$$\frac{dx}{dt} + A(t)x = f(t, x), \tag{1}$$

wobei $x(t)$ eine für gewisse reelle t erklärte Funktion mit Werten aus einem B-Raum E ist, ferner $A(t)$ einen linearen unbeschränkten abgeschlossenen Operator mit Definitionsbereich $D(A)$ und $f(t, x)$ einen beschränkten nichtlinearen Operator bedeutet. Krasnosel'skiĭ, Krein und Verfasser [dieselben Dokl. (N.S.) **111** (1956), 19–22; **112** (1957), 990–993; MR **19**, 747] haben Existenzsätze für (1) bewiesen. Jetzt betrachtet Verfasser die Abhängigkeit der Lösung von A, f und dem Anfangswert $x(s) = x^0$, wobei er ähnliche Resultate wie bei üblichen Differentialgleichungen erzielt (zum Beispiel benützt er Lipschitzbedingungen bei f). Die Betrachtungen beruhen teilweise darauf, daß im homogenen Fall ($f \equiv 0$) die Lösung x mit Hilfe eines stetigen Operators $U(t, s)$ dargestellt werden kann in der Form $x(t) = U(t, s)x^0$, und daß (1) im wesentlichen äquivalent ist mit der Integralgleichung $x(t) = U(t, 0)x^0 + \int_0^t U(t, s)f(s, x(s))ds$. Als Spezialfall betrachtet er Näherungsverfahren für (1), die sich durch Anwendung von Projektionsoperatoren ergeben (wobei E ein Hilbertraum sein soll). *K. Zeller* (Tübingen)

1051:

Nevanlinna, Rolf. Zur Theorie der Normalsysteme von gewöhnlichen Differentialgleichungen. Rev. Math. Pures Appl. **2** (1957), 423–428.

The concern is with the existence and uniqueness of the solution, for which $\mathbf{y}(x_0) = \mathbf{y}_0$, if

$$\frac{d\mathbf{y}}{dx} = \mathbf{f}(x, \mathbf{y}),$$

where x is a real variable and $\mathbf{y}$ is a vector variable belonging to a linear space, of any dimension, finite or infinite, in which the metric is that of Minkowski-Banach. A set of approximations is constructed by Cauchy's polygon method, and its convergence to a solution is proved provided $\int d\rho/\phi(\rho)$ diverges, where $\phi(\rho) = \sup |f(x, \mathbf{y}_1) - f(x, \mathbf{y}_2)|$ for $|\mathbf{y}_1 - \mathbf{y}_2| \le \rho$, and $|\mathbf{y}|$ denotes the norm of $\mathbf{y}$. The same condition ensures uniqueness. The treatment is clear and compact, and the symbolism is independent of the dimension of the space.

T. M. Cherry (Melbourne)

PARTIAL DIFFERENTIAL EQUATIONS

See also 1008, 1115, 1116, 1226, 1332, 1420, 1503, 1548.

1052:

Hornich, Hans. Equazioni differenziali parziali lineari dappertutto non risolubili. Rend. Sem. Mat. Fis. Milano **26** (1954/55), 3–8 (1957).

L'A. mostra, con un esempio, che l'equazione

$$\frac{\partial u}{\partial x} + \varphi(x, y)\frac{\partial u}{\partial y} = f(x, y)$$

può non avere soluzioni in nessuna parte aperta di una regione G. Questo avviene, naturalmente, se la funzione $\varphi(x, y)$ si suppone soltanto continua; l'A. prova che si può scegliere la $\varphi(x, y)$ in modo che l'equazione delle caratteristiche $y' = \varphi(x, y)$ abbia "doppie vie" in numero infinito e dappertutto dense in G; di qui si deduce immediatamente la tesi.

L. Amerio (Milano)

1053:

Ladyženskaya, O. A. On the construction of discontinuous solutions of quasi-linear hyperbolic equations as limits of solutions of the corresponding parabolic equations when the "coefficient of viscosity" tends toward zero. Trudy Moskov. Mat. Obšč. **6** (1957), 465–480. (Russian)

In einer Reihe von Untersuchungen, die man E. Hopf, O. A. Oleĭnik, P. Lax, A. H. Tychonoff und A. A. Samarski verdankt [cf., e.g., E. Hopf, Comm. Pure Appl. Math. 3 (1950), 201–230; MR 13, 846; P. Lax, Contributions to the theory of partial differential equations, Princeton, 1954, pp. 211–229; Comm. Pure Appl. Math. 7 (1954), 159–193; MR 16, 828, 524], wurde versucht für das Cauchysche Anfangswertsproblem

$$(1)\qquad \frac{\partial u}{\partial t}+\frac{\partial \varphi(x,t,u)}{\partial x}=0,\quad u|_{t=0}=\psi(x)$$

Lösungen innerhalb beliebig großer Bereiche (x, t) zu konstruieren. Selbst für erheblich glatte Funktionen φ, ψ ist die Existenz von eindeutigen und stetigen Lösungen für $t\geqq 0$ nicht unbedingt garantiert, da sich die Projektionen der zugehörigen Charakteristiken auf die (x, t)-Ebene für $t>0$ überschneiden können, was die Konstruktion der gewünschten Lösungen verhindert. Indessen bieten bisweilen physikalische oder geometrische Zusatzannahmen einen Ausweg, wie etwa das Prinzip der "verschwindenden Viskosität". An Stelle von (1) wird dann

$$(2)\qquad \frac{\partial u}{\partial t}+\frac{\partial \varphi(u)}{\partial x}=\varepsilon\frac{\partial^2 u}{\partial x^2}$$

behandelt und die Lösung des neuen Cauchyschen Problems (2), $u^\varepsilon(x, t)$, durch Grenzübergang $\varepsilon\to 0$ für die Lösung des alten Problems (1) verwertet. Dabei hatte sich in den Untersuchungen von E. Hopf und in der Dissertation von Oleĭnik gezeigt, daß dieser Grenzübergang $\lim_{\varepsilon\to 0} u^\varepsilon(x, t)=u(x, t)$ zu einer unstetigen Lösung führt. Der Verfasserin gelingt eine einfachere Begründung des "Prinzips der verschwindenden Viskosität". Die Gesamtheit aller Lösungen $u^\varepsilon(x, t)$ strebt gegen eine einzige meßbare und beschränkte Funktion $u(x, t)$, welche die verallgemeinerte Lösung des ursprünglichen Problems darstellt. Für $u(x, t)$ gilt die Integralrelation

$$\iint_{t>0}\left[u\frac{\partial\Phi}{\partial t}+\varphi(u)\frac{\partial\Phi}{\partial t}\right]dxdt+\int_{-\infty}^{+\infty}\varphi(x)\Phi(x,0)dx=0$$

und die Ungleichung

$$\frac{1}{\Delta}[u(x+\Delta, t)-u(x, t)]\leqq M(t).$$

Dabei bedeutet $\Phi(x, t)$ eine beliebige stetig differenzierbare und beschränkte Funktion ihrer Argumente und $M(t)$ eine auf $(0, T]$ stetige Funktion. Schließlich kann auch noch die Eindeutigkeit der Lösungsfunktion unter den angegebenen Bedingungen gezeigt werden. Der Beweisführung werden in § 1 eine Reihe von Hilfssätzen vorausgeschickt. Der Nachweis des Eindeutigkeitssatzes findet sich auch schon bei O. A. Oleĭniks Untersuchungen [Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 1098–1101; MR 18, 656]. *M. Pinl* (Köln)

1054:

Oleĭnik, O. A.; and Vvedenskaya, N. D. The solution of Cauchy problem and boundary value problem for nonlinear equations in the discontinuous functions class. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 503–506. (Russian)

Es handelt sich um die Behandlung des Cauchyschen Anfangswertproblems der quasilinearen partiellen Differentialgleichung

$$(1)\qquad \frac{\partial u}{\partial t}+\frac{\partial\varphi(t,x,u(t,x))}{\partial x}+\psi(t,x,u(t,x))=0,$$

welche aus

$$(2)\qquad \varepsilon\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}+\frac{\partial\varphi}{\partial x}+\psi$$

durch den Grenzübergang $\varepsilon\to 0$ hervorgeht. — Verfasser sprechen von einer Lösung $u(t, x)$ des verallgemeinerten Cauchyschen Problems der Differentialgleichung (1) im Gebiet $G\{0\leqq t\leqq T, -\infty<x<+\infty\}$ mit der Anfangsbedingung $u(0, x)=u_0(x)$, wenn für diese Funktion und eine beliebige beschränkte glatte Funktion $f(t, x)$, die für $t\geqq T$ verschwindet, die Gleichung

$$\iint_{(G)}\left[u\frac{\partial f}{\partial t}+\varphi(t,x,u)\frac{\partial f}{\partial x}-\psi(t,x,u)f\right]dxdt+\int_{-\infty}^{+\infty}f(0,x)u_0dx=0$$

gilt, und wenn ferner für $0\leqq t\leqq T$ und $-\infty<x_i<+\infty$, $i=1, 2$, eine stetige Funktion $K(t, x_1, x_2)$ existiert, sodaß für alle (t_1, x_1), (t, x_2) aus G für $t>0$

$$[u(t, x_1)-u(t, x_2)](x_1-x_2)^{-1}\leqq K(t, x_1, x_2)$$

besteht. Dann wird gezeigt: wenn es eine solche verallgemeinerte Lösung des Cauchyschen Problems gibt, dann nur eine einzige. Weiterhin wird die Existenz der Lösung bewiesen und ihre Abhängigkeit von der vorgegebenen Funktion $u_0(x)$ untersucht. Erklärt man analog ein verallgemeinertes Cauchysches Problem für die Differentialgleichung (2), so geht für $\varepsilon\to 0$ eine Lösung u_ε dieses Problems über in die des ursprünglich definierten Cauchyschen Problems. Anschließend wird mit den Bedingungen

$$u(0, x)=u_0(x),\ u(t, 0)=u_1(t),\ u(t, 1)=u_2(t)$$

(u_0, u_1, u_2 beschränkte stetige Funktionen) ein Randwertproblem gestellt und dessen Lösung durch weitere Bedingungen definiert. Wieder lassen sich Eindeutigkeits- und Existenzsätze aufstellen. *M. Pinl* (Köln)

1055:

Oleĭnik, O. A. Discontinuous solutions of non-linear differential equations. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 3–73. (Russian)

Diese inhaltsreiche Arbeit enthält nicht weniger als 22 Sätze ergänzt durch 23 Hilfssätze. Die Untersuchung beginnt mit der Formulierung des Cauchyschen Problems der quasilinearen partiellen Differentialgleichung

$$(1)\qquad \frac{\partial u}{\partial t}+\frac{\partial\varphi(t,x,u)}{\partial x}+\psi(t,x,u)=0$$

und der Untersuchung der Eindeutigkeit der Problemlösung. Dabei werden die Funktionen φ und ψ eindeutig und in allen ihren Argumenten stetig innerhalb $-\infty<x<+\infty$ und $0\leqq t\leqq T\leqq\infty$ vorausgesetzt. Sodann werden bestimmte Verallgemeinerungen von Lösungen des Cauchyschen Problems der Gleichung (1) angegeben und die Bedingungen der Eindeutigkeit auch solcher Lösungen erörtert. Dabei nennt Verfasser eine beschränkte meßbare Funktion $u(t, x)$ eine verallgemeinerte Lösung des Cauchyschen Problems der Differentialgleichung (1) im Gebiet $G(-\infty<x<+\infty, 0\leqq t\leqq T\leqq\infty)$ zur — für alle x beschränkt und meßbar vorgegebenen — Anfangsbedingung

(2) $u(0, x)=u_0(x), \quad -\infty<x<+\infty,$

wenn (i) für jede in G stetig differenzierbare Funktion $f(t, x)$, welche für $t=T$ und außerhalb eines bestimmten endlichen Bereichs verschwindet, die Integralbeziehung

$$\iint_{(G)}\left[\frac{\partial f}{\partial t}u(t, x)+\frac{\partial f}{\partial x}\varphi(t, x, u(t, x))-f\psi(t, x, u)\right]dxdt + \int_{-\infty}^{+\infty}f(0, x)u_0(x)dx=0$$

gilt und wenn (ii) für $0<t<T$, $-\infty<x_1<+\infty$, $-\infty<x_2<+\infty$ eine stetige Funktion $K(t, x_1, x_2)$ existiert, derart daß für alle Punkte (t, x_1) aus G die Beziehung

$$\frac{u(t, x_1)-u(t, x_2)}{x_1-x_2}\leqq K(t, x_1, x_2)$$

besteht. Unter diesen Voraussetzungen gilt der Eindeutigkeitssatz: die verallgemeinerte Lösung des Cauchyschen Problems (1) zur Anfangsbedingung (2) ist eindeutig. Analog wie dieses Ergebnis läßt sich darüber hinaus beweisen: zwei verallgemeinerte Lösungen $u_1(t, x)$ und $u_2(t, x)$ der Gleichung (1), welche in den Gebieten D_1 und D_2 definiert sind und sich im Intervall $[a, b]$ für $t=0$ auf $u_0(x)$ reduzieren, fallen in jedem Bereich $D_{\tau\alpha}$ zusammen, der durch die Geraden

$$t=0,\ x-a-\alpha=At,\ x-b+\alpha=-At,\ t=\tau$$

begrenzt wird, wofern $D_{\tau\alpha}$ im Durchschnitt der Gebiete D_1 und D_2 liegt. — Dem Eindeutigkeitsbeweise folgt der Existenzbeweis der verallgemeinerten Lösungen des Cauchyschen Problems nach der "Methode der beschränkten Differenzen". Dabei kommt es zunächst auf die Konstruktion geeigneter Funktionenfolgen U_{hl}^{i} an, deren Grenzverhalten gegenüber der Lösung $u(x, t)$ eingehend untersucht wird. Insbesondere gilt unter entsprechenden Voraussetzungen, für $i\to\infty$,

$$\int_{-X}^{+X}|U_{hl}^{i}(t, x)-u(t, x)|dx\to 0,$$

$$\int_0^T\int_{-X}^{+X}|U_{hl}^{i}(t, x)-u(t, x)|dxdt\to 0.$$

In einem weiteren Abschnitt wird das Cauchysche Problem für quasilineare parabolische Differentialgleichungen erörtert und insbesondere der Fall, daß sich die verallgemeinerte Lösung des Cauchyschen Problems einer nichtlinearen Differentialgleichung erster Ordnung als Grenzfall der entsprechenden Lösung des gleichen Problems für eine parabolische Differentialgleichung auffassen läßt. Dann handelt es sich um die Differentialgleichung

$$(3)\quad \varepsilon\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}+\frac{\partial\varphi(t, x, u)}{\partial x}+\psi(t, x, u),\ u(0, x)=u_0(x)$$

Dabei bedeutet ε einen kleinen Parameter. Dann gilt: im Bereich G gibt es eine Funktion $u_\varepsilon(x, t)$ mit allen stetigen Ableitungen, die in (3) eingehen. Diese Funktion $u_\varepsilon(t, x)$ befriedigt (3) mit dem Anfangswert $u(0, x)=u_0(x)$ im folgenden Sinn: für jede beliebige stetige Funktion $f(t, x)$ die für hinreichend große Werte x verschwindet, gilt

$$\int_{-\infty}^{+\infty}[f(t, x)u_\varepsilon(t, x)-f(0, x)u_0(x)]dx\to 0,\ t\to 0.$$

Wenn $u_0(x)$ für $x=x_1$ stetig ist, gilt $\lim_{t\to 0, x\to x_1}u_\varepsilon(t, x)=u_0(x_1)$. Jede beschränkte Lösung $u_\varepsilon(t, x)$ des Cauchyschen Problems (3) mit der Anfangsbedingung $u(0, x)=u_0(x)$ ist eindeutige Lösung. Für $\varepsilon\to 0$ geht $u_\varepsilon(t, x)\to u(t, x)$, d.h. gegen die Lösung des Cauchyschen Problems (1) mit $u(0, x)=u_0(x)$ und $-\infty<x<+\infty$, sofern

$$\int_{-X}^{+X}|u_\varepsilon(t, x)-u(t, x)|dx\to 0,\ \varepsilon\to 0.$$

Ist dann $u(t, x)$ eine nach der Methode der beschränkten Differenzen konstruierte Lösung des Cauchyschen Problems (1) mit $u(0, x)=u_0(x)$ und $-\infty<x<+\infty$ und ist (t_0, x_0) ein Punkt aus dem Gebiet G, derart, daß $u(t, x)$ in (t_0, x_0) stetig bleibt, so gilt

$$u_\varepsilon(t_0, x_0)\to u(t_0, x_0),\ \varepsilon\to 0.$$

Sind $u(t, x)$ und $v(t, x)$ zwei verallgemeinerte Lösungen des Cauchyschen Problems (1) aus dem Gebiet G und gilt

$$u(0, x)=u_0(x),\ v(0, x)=v_0(x),\ -\infty<x<+\infty,$$

wobei $u_0(x)$ und $v_0(x)$ meßbare Funktionen sind, die unter der Schranke m bleiben, so gilt

$$\int_{a_1}^{a_2}|u(t, x)-v(t, x)|dx<\delta e^{Bt},$$

wofern $\int_{a_1-At}^{a_2+At}|u_0(x)-v_0(x)|dx<\delta$. Dabei gilt $a_1, a_2, \delta>0$, $a_1<a_2$, $A=\max_\Omega|\varphi_u|$, $B=\max_\Omega|\psi_u|$ und das Gebiet Ω wird durch die Bedingungen

$$\{(t, x)\subset G,\ |u|\leqq M\},\ M=\sup_{(G)}\{|u(t, x), v(t, x)|\}$$

bestimmt. Ist $u^n(x, t)$ eine verallgemeinerte Lösung des Cauchyschen Problems (1) im Gebiet G mit der Anfangsbedingung $u^n(0, x)=u_0^n(x)$ und $|u_0^n(x)|\leqq m$ $(n=1, 2, \cdots)$, und besteht

$$\int_{-\infty}^{+\infty}f(x)[u_0^n(x)-u_0(x)]dx\to 0$$

für $n\to\infty$ und beliebige stetige Funktionen $f(x)$, von denen nur $f(x)\to 0$ verlangt wird, wenn x hinreichend groß wird, so gilt

$$u^n(t, x)\to u(t, x),\ n\to\infty,$$

wobei $u(t, x)$ die verallgemeinerte Lösung des Cauchyschen Problems (1), (2) darstellt in dem Sinne, daß für beliebige beschränkte Teilbereiche G_1 aus G

$$\iint_{(G_1)}|u^n(t, x)-u(t, x)|dxdt\to 0$$

gilt für $n\to\infty$. Schließlich wird von der verallgemeinerten Lösung $u(t, x)$ des Cauchyschen Problems (1), (2) das Bestehen der Beziehung

$$\int_{-\infty}^{+\infty}f(x)[u(t, x)-u_0(x)]dx\to 0,\ t\to 0$$

bewiesen, wobei von der Funktion $f(x)$ wiederum nur Stetigkeit und Abklingen gegen Null verlangt wird mit $x\to\infty$. Ein letzter Abschnitt behandelt Grenzfunktionen $u_\varepsilon(t, x)$ zur Konstruktion von Lösungen von Randwertproblemen parabolischer Differentialgleichungen und den Grenzübergang $\varepsilon\to 0$ in diesem Falle. Eine schon an sich wertvolle Bibliographie umfasst 29 meist russisch geschriebene Abhandlungen der letzten 18 Jahre.

M. Pinl (Köln)

1056:

Oleĭnik, O. A. On the equations of unsteady filtration. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 1210–1213. (Russian)

In the present paper the author presents uniqueness and existence proofs for the solutions of the Cauchy-problem for the non-linear partial differential equation (1) $\partial u/\partial t=\partial^2\varphi(t, x, u)/\partial x^2$, $u(0, x)=u_0(x)$. The function $u_0(x)\geqq 0$ and bounded for all x; $\varphi_u'\geqq 0$ for $u\geqq 0$ and $\varphi_u'>0$ for $u>0$. When $\varphi=u^n$, then (1) describes filtration pro-

cesses. The procedure employed by the author is based on some integral relations, auxiliary functions, and certain other quantities.

{It seems to the present reviewer that a good deal of the analysis (including some essential points) is suppressed. As a matter of fact, in view of these omissions, the author's results cannot be considered conclusive. Also, the details are not found in the references cited by the author. Non-linear filtration processes are, unfortunately, very complicated from the mathematical point of view. And, therefore, rigorous proofs of the Cauchy-problem will be valuable; particularly, for short time intervals.}

K. Bhagwandin (Oslo)

1057:

Oleĭnik, O. A. On the uniqueness of the generalized solution of the Cauchy problem for a non-linear system of equations occurring in mechanics. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 6(78), 169–176. (Russian)

In der Halbebene $t\geqq 0$ sei D ein von der Geraden $t=0$ und einer glatten Kurve Γ begrenztes Gebiet. $u_0(x)$ und $v_0(x)$ seien im Intervall $[a, b]$ beschränkte meßbare Funktionen. Die im Gebiet D erklärten Funktionen $u(t, x)$ und $v(t, x)$ seien Lösungen des hyperbolischen Systems

$$(1)\qquad \frac{\partial u}{\partial t}+\frac{\partial \varphi(t, x, v)}{\partial x}=0,\quad \frac{\partial v}{\partial t}-\frac{\partial u}{\partial x}=0$$

mit den Anfangsbedingungen

$$(2)\qquad u|_{t=0}=u_0(x),\quad v|_{t=0}=v_0(x)$$

im folgenden Sinn: sind $f(t, x)$, $\psi(t, x)$ zwei beliebige stetig differenzierbare Funktionen welche in der Umgebung der Kurve Γ verschwinden, so gelten die Gleichungen

$$(3)\qquad \begin{aligned}&\iint_{(D)}\Big[\frac{\partial f}{\partial t}u+\frac{\partial f}{\partial x}\varphi(t, x, v)\Big]dxdt+\int_a^b u_0(x)f(0, x)dx=0,\\ &\iint_{(D)}\Big[\frac{\partial \psi}{\partial t}v-\frac{\partial \psi}{\partial x}u\Big]dxdt+\int_a^b v_0(x)\psi(0, x)dx=0.\end{aligned}$$

Im Gebiet D wird ferner noch das Bestehen der Ungleichung

$$(4)\qquad u(t, x_1)-u(t, x_2)<K(x_1-x_2),$$

mit einer Konstanten K angenommen (so daß auch $u(t, x+0)\leqq u(t, x-0)$ gilt). Dann wird bewiesen: genügen die in D definierten beschränkten und meßbaren Funktionenpaare $u_1(t, x)$, $v_1(t, x)$ und $u_2(t, x)$, $v_2(t, x)$ dem System (1), den Anfangsbedingungen (2) — im Sinne von (3) — und der Zusatzbedingung (4), so gilt $u_1=u_2$, $v_1=v_2$ in einem Gebiet D_τ, das von den Geraden

$$t=0,\ t=\tau,\ x-a=At,\ x-b=-At$$

begrenzt wird mit

$$A=\max_D\sqrt{(-\Phi(t, x))},\ \Phi(t, x)=\frac{\varphi(t, x, v_1)-\varphi(t, x, v_2)}{v_1-v_2},$$

wobei $\tau>0$ beliebig ist, solange D_τ zu D gehört. — Ein zweiter Satz lautet: es gibt im Streifen $0\leqq t\leqq T$ ein und nur ein Paar Funktionen u, v, welche das System (1) mit den Anfangsbedingungen (2) für stückweise stetige Anfangsfunktionen $u_0(x)$ und $v_0(x)$ befriedigen, wofern für alle Punkte (t_1, x) und (t_2, x) und $0\leqq t\leqq T$

$$v(t_1, x)-v(t_2, x)<\tilde{K}(t_1, t_2, x)(t_1-t_2)$$

gilt mit einer bis auf endlich viele Punkte ihres Definitionsgebietes überall stetigen und beschränkten Funktion $\tilde{K}$.

M. Pinl (Köln)

1058:

Maslennikova, V. N. Solution in explicit form of the Cauchy problem for a system of partial differential equations. Izv. Akad. Nauk SSSR. Ser. Mat. **22** (1958), 135–160. (Russian)

The initial-value problems in unbounded xyz-space for the system

$$(1)\qquad \frac{\partial \mathbf{v}}{\partial t}-\mathbf{v}\times\mathbf{k}+\operatorname{grad} p=\mathbf{F},\quad \alpha^2\frac{\partial p}{\partial t}+\operatorname{div}\mathbf{v}=\psi,$$

and for the equation

$$(2)\qquad Lu\equiv\frac{\partial^2\Delta u}{\partial t^2}+\frac{\partial^2 u}{\partial z^2}-\alpha^2\Big(\frac{\partial^4 u}{\partial t^4}+\frac{\partial^2 u}{\partial t^2}\Big)=f,$$

are solved explicitly by means of integral formulas of Green's type. The fundamental solution of (1), which is constructed from certain particular solutions $\Phi(x, y, z, t; x_0, y_0, z_0, t_0; \alpha)$ of $L\Phi\equiv 0$, has the form

$$\mathbf{v}=\operatorname{grad}\frac{\partial^2\Phi}{\partial t^2}+\Big(\operatorname{grad}\frac{\partial\Phi}{\partial t}\times\mathbf{k}\Big)+\mathbf{k}(\operatorname{grad}\Phi, \mathbf{k}),$$

$$p=-\frac{\partial^3\Phi}{\partial t^3}-\frac{\partial\Phi}{\partial t}.$$

The results and the methods for both (1) and (2) are direct generalizations of those of S. L. Sobolev [same Izv. **18** (1954), 3–50; MR **16**, 1029], who considered the respective cases for $\alpha=0$.

R. N. Goss (San Diego, Calif.)

1059:

Perčinkova, D. Formation d'une classe d'équations aux dérivées partielles. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **8** (1955), 51–67. (Serbo-Croatian. French summary)

On considère, dans le cas le plus général, le problème de la formation des équations aux dérivées partielles par élimination des fonctions F_k de la relation (1) $z=\sum_{k=1}^{n}F_k(a_kX+b_kY)$, et des fonctions f_k de la relation (2) $z=\prod_{k=1}^{n}f_k(a_kX+b_kY)$ (a_k, b_k=const; $X=X(x)$, $Y=Y(y)$).

D. S. Mitrinovitch (Belgrade)

1060:

Bojanić, R. Remarque sur un procédé de formation des équations aux dérivées partielles. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **8** (1955), 71–73. (Serbo-Croatian. French summary)

(Voir le rapport précédent.) On indique une simplification dans la formation d'une équation aux dérivées partielles dont la solution est la fonction, définie par (2).

D. S. Mitrinovitch (Belgrade)

1061:

Gel'fand, I. M.; and Šapiro, Z. Ya. Homogeneous functions and their extensions. Amer. Math. Soc. Transl. (2) **8** (1958), 21–85.

Translated from Uspehi Mat. Nauk (N.S.) **10** (1955), no. 3(65), 3–70 [MR **17**, 371].

1062:

Kreyszig, Erwin. Über ein System partieller Differentialgleichungen. Arch. Rational Mech. Anal. **1** (1957), 46–53.

The class of systems of partial differential equations considered is of the form $U_{x_\nu x_\nu}+U_{y_\nu y_\nu}+\varphi_\nu(x_\nu, y_\nu)U=0$, $\nu=1, 2$, or — after introduction of conjugate coordinates $z_\nu=x_\nu+iy_\nu$, $z_\nu^*=x_\nu-iy_\nu$ — (*) $U^*_{z_\nu z_\nu}+F_\nu(z_\nu, z_\nu^*)U=0$. The author shows that if the functions F_ν are of a certain type, the system (*) has an infinite set of particular

solutions with the following properties: 1) Each of these solutions satisfies ordinary differential equations (with algebraic coefficients) in each of the four variables, where the order of the equations depends only on F_1 and F_2 and not on the particular solution; 2) the set of these solutions is dense in the class of solutions of (*).

Z. Nehari (Pittsburgh, Pa.)

1063:

Bachelis, R. D. On analytical solutions of differential equations in partial derivatives of the third order. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 567–570. (Russian)

The author establishes conditions under which the partial differential equation of third order

$$\Phi(x_1, x_2, x_3, v, \nabla v, \nabla^2 v, \nabla^3 v)=0,$$

with Φ analytic in its arguments, has a solution analytic in the neighborhood of the origin in the (x_1, x_2, x_3)-space, satisfying the boundary conditions $v=0$ for $x_1=0$, or $x_2=0$, or $x_3=0$.

F. Browder (New Haven, Conn.)

1064:

Browder, Felix E. Eigenfunction expansions for nonsymmetric partial differential operators. I. Amer. J. Math. **80** (1958), 365–381.

A densely defined operator A in a Hilbert space H is said to be subnormal if there exists a Hilbert space H_1 containing H as subspace and a normal operator A_1 in H_1 such that $A \subseteq A_1$. If A is subnormal in H, and $A \subseteq A_1$ as above, let E be the resolution of the identity of A_1 in H_1. If P is the projection of H_1 onto H, the generalized resolution of the identity F corresponding to A_1 is defined by $F=PE$. The Hilbert space considered in this work is obtained from $L^2(G)$, where G is an open set in R_n, in the following way. Let B be a linear operator with $C_c^\infty(G)$ as its domain, and range in the space of locally square-summable functions on G. It is assumed that there exist $\phi \in C^0(G)$, $N \in C^r(G)$ for some $r \geqq 0$, with $\phi(x)>0$, $N(x)>0$ for x in G such that: (i) for ϕ, $\psi \in C_c^\infty(G)$, $(B\phi, \psi)=(\phi, B\psi)$, $(B\phi, \phi) \geqq (\phi\phi, \phi)$; (ii) $(B\phi, \phi) \leq \sum_{|\alpha| \leq r} (ND^\alpha\phi, D^\alpha\phi)$, $\phi \in C_c^\infty(G)$; (iii) $(\phi^{-1}B\phi, B\phi) < \infty$ for $\phi \in C_c^\infty(G)$. On $C_c^\infty(G)$ the inner product $[\phi, \psi]=(B\phi, \psi)$ is defined, and H is the completion of $C_c^\infty(G)$ with respect to the norm $\|u\|_H=[u, u]^{\frac{1}{2}}$. It is proved that if A is subnormal in this H, then there exists an eigenfunction expansion corresponding to every generalized resolution of the identity of A. The proof uses a differentiation theorem due to G. Birkhoff and I. Gelfand. If L is a linear partial differential operator with C^∞ coefficients, and B is a positive differential operator, and if L has a subnormal realization in H, there exists an eigenfunction expansion, which is essentially in terms of the eigenfunctions of $B^{-1}L$. For the case $B=\phi(x)I$, $\phi \in C_c^\infty(G)$, the eigenfunctions are distribution solutions of $(L-\rho\phi)u=0$. If in addition L is hypoelliptic, the eigenfunctions are in $C^\infty(G)$.

E. A. Coddington (Los Angeles, Calif.)

1065:

***Titchmarsh, E. C. Eigenfunction expansions associated with second-order differential equations. Vol. 2.** Oxford, at the Clarendon Press, 1958. xi+404 pp. $11.20.

The object of this book is the consideration of the expansion of an arbitrary function in terms of the eigenfunctions of the differential equation $\Delta\psi+(\lambda-q)\psi=0$, where Δ is Laplace's operator in space of k dimensions and q is a function of position in that space. It is the continuation of a book of the same title which appeared in 1946 [MR 8, 458] and which dealt with the case $k=1$. The present work, though more difficult, is of the same general character, making use of complex variable technique, and "is the result of an attempt by an 'analyst' to understand those parts of quantum mechanics which can be regarded as exercises in analysis." Much of the book has appeared in Professor Titchmarsh's published papers.

The chapters of the book number on from the 1946 volume. In chapter XI the two-dimensional case is discussed, the region being a finite rectangle on whose boundary the eigenfunctions $\psi_n(\mathbf{x})$ are required to vanish. In this case, the spectrum is discrete; if $f(\mathbf{x})$ and $\Delta f-qf$ are L^2 over the rectangle, $f(\mathbf{x})$ has an absolutely and uniformly convergent expansion as a series of eigenfunctions. The fundamental tool is the Green's function.

In chapter XII the theory is extended to the case when the region is the whole plane. If the Green's function for the square $-X \leqq x \leqq X$, $-X \leqq y \leqq X$ is $G_X(\mathbf{x}, \boldsymbol{\xi}, \lambda)$, the Green's function for the whole plane is the limit of G_X as $X \to \infty$ through an appropriate sequence of values. It is not obvious from the proof, which depends on the theory of compact sets of functions, whether the Green's function so obtained is unique or not; but it is shown that if $q(\mathbf{x}) \geqq -Q(r)$, where $r=|\mathbf{x}|$, and $Q(r)$ is a positive non-decreasing function of r, the Green's function is unique if $\int^\infty dr/\sqrt{Q(r)}$ is divergent. If

$$H(\mathbf{x}, \boldsymbol{\xi}, \mu)=\lim_{\nu \to 0} \int_0^\mu \operatorname{im} G(\mathbf{x}, \boldsymbol{\xi}, \sigma+i\nu)d\sigma,$$

the expansion theorem takes the form

$$f(\mathbf{x})=\frac{1}{\pi}\lim_{T \to \infty} \int_{-\infty}^{\infty} [H(\mathbf{x}, \boldsymbol{\xi}, T)-H(\mathbf{x}, \boldsymbol{\xi}, -T)]f(\boldsymbol{\xi})d\boldsymbol{\xi}.$$

The nature of the spectrum depends on the properties of the function H. Various extensions of the theory, in particular to expansions in space of more than two dimensions, are discussed in chapter XIII. The manner in which the eigenvalues vary as q or as the region is varied are discussed in chapter XIV.

Practically the only cases where explicit solutions of the expansion problem can be obtained are those in which the variables are separable. This happens, for instance, in the case of two dimensions $(\mathbf{x}=(x, y))$, if $q=h(x)+j(y)$. Then if

$$\frac{d^2u}{dx^2}+(\mu-h(x))u=0, \quad \frac{d^2v}{dy^2}+(\nu-j(y))v=0$$

have eigenvalues μ_m and ν_n, the equation $\Delta\psi+(\lambda-h(x)-j(y))\psi=0$ has eigenvalues $\mu_m+\nu_n$. Chapter XV is devoted to proving that all eigenvalues are of this form.

The nature of the spectrum depends on the properties of $q(\mathbf{x})$. A complete classification is not made, but sufficient conditions to ensure that it is discrete are found in chapter XVI, and theorems on the distribution of eigenvalues in the discrete case are given in chapter XVII.

The expansion theorems in the earlier part of the book require heavy restrictions on the function $f(\mathbf{x})$ to be expanded. The object of chapter XVIII is to prove convergence or summability theorems under weaker conditions, such as that $f(\mathbf{x})$ is L^2 and satisfies some local condition, it being assumed that the spectrum is discrete and bounded below.

Perturbation theory is discussed is chapter XIX and XX, and the special problems, which arise when $q(x, y, \cdots)$ is periodic in each of its arguments, are considered in chapter XXI. The book concludes with a valuable chapter of 62 pages containing proofs of a number of theorems which are quoted in the text but have no specific con-

nexion with eigenfunction theory. There is a long list of works on the field covered by both parts of the book.

The book is not easy reading, as the subject "is pursued without much regard to the interests of theoretical physicists." But it is a valuable addition to the literature of this field. *E. T. Copson* (St. Andrews)

1066:

Garnir, H. G.; et Gobert, J. Le problème de Dirichlet-Neumann pour les opérateurs métaharmonique, des ondes et de la diffusion par la méthode des fonctions propres. I, II. Bull. Soc. Roy. Sci. Liège **26** (1957), 279–289; **27** (1958), 17–27.

In queste due Note gli A. riescono a dare una trattazione veramente semplice dei problemi generalizzati di Dirichlet-Neumann per gli operatori metaarmonico, delle onde e della diffusione, mediante il classico metodo di sviluppi in serie di autosoluzioni. Nonostante la brevità delle Note e l'impostazione moderna in termini di soluzioni deboli, la trattazione è completa ed elementare e presuppone solo le conoscenze fondamentali sugli spazi di Hilbert, richiamando in modo opportuno le nozioni occorrenti.

Sia Ω un aperto limitato dello spazio euclideo a n dimensioni E^n e sia Γ_1 una porzione della frontiera Γ di Ω e $\Gamma_2=\Gamma-\Gamma_1$. Indichiamo con N lo spazio delle funzioni u: 1) tali che u, $\partial u/\partial x_1$, $\cdots$, $\partial u/\partial x_n$, e Δu (Δ laplaciano) appartengano a $L^2(\Omega)$, la derivazione essendo intesa nel senso delle distribuzioni; 2) "nulle" su Γ_1, nel senso che esista una successione di funzioni v_m appartenenti con le derivate prime a $L^2(\Omega)$ e nulle identicamente in un intorno di Γ_1 (variabile con m e tendente a Γ_1) tale che $v_m\to u$, $\partial v_m/\partial x_i\to\partial u/\partial x_i$ ($i=1, \cdots, n$) in $L^2(\Omega)$; 3) aventi "derivata normale nulla" su Γ_2 nel senso che

$$(\Delta u, v)+\sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\right)=0$$

per ogni v soddisfacente alla condizione 2), $(\ ,\ )$ essendo il prodotto scalare in $L^2(\Omega)$.

Supponiamo inoltre che Γ_1 sia sufficientemente regolare, in modo che esista una successione di autovalori z_k negativi, tendenti a $-\infty$ e di autosoluzioni reali u_k del laplaciano per il problema di Dirichlet-Neumann ($\Delta u_k=z_k u_k$, $u_k\in N$), constituenti un sistema ortonormale e completo in $L^2(\Omega)$. Sia poi per ogni z complesso L_z^2 il sottospazio di $L^2(\Omega)$ ortogonale a tutte le eventuali autosoluzioni di autovalore z. Il problema di Dirichlet-Neumann per l'operatore $-\Delta+z$ consiste allora nella ricerca di $u\in N\cap L_z^2$ tale che $-\Delta u+zu=f$, f essendo dato in L_z^2. Nelle ipotesi fatte, gli A. dimostrano che esiste una e una sola soluzione per ogni $f\in L_z^2$ data da

$$G_z f=\sum_{z_k\neq z}\frac{(f, u_k)}{z-z_k}u_k$$

e studiano le proprietà dell'operatore di Green G_z del problema.

Il problema di Dirichlet-Neumann per l'equazione

$$(-\Delta+a(\partial^2/\partial t^2)+b(\partial/\partial t)+c)u=m$$

(delle onde se $a>0$ e della diffusione se $a=0$ e $b>0$) nel cilindro $\Omega\times]0, \infty[$ é impostato nel modo seguente: Indicato con $\mathcal{D}(E^1)$ lo spazio delle funzioni $\varphi(t)$ indefinitamente differenziabili a supporto compatto in E^1, si cerca $u[\varphi]$ distribuzione vettoriale a valori in $L^2(\Omega)$ e "nulla" sotto lo zero, in modi che tali siano anche le $\partial u/\partial x_i$ ($i=1, \cdots, n$) e Δu e che $u[\varphi]\in N$ e

$$u\left[a\frac{\partial^2\varphi}{\partial t^2}-b\frac{\partial\varphi}{\partial t}+c\varphi\right]-\Delta u[\varphi]=$$
$$\int_0^\infty m(x, t)\varphi(t)dt+(au_1+bu_0)\varphi(0)-au_0\frac{\partial\varphi(0)}{\partial t}$$

per ogni $\varphi\in\mathcal{D}(E^1)$; $m(x, t)$ è una funzione data in $\Omega\times]-\infty, \infty[$, nulla in $\Omega\times]-\infty, 0[$ e che appartiene a $L^2(\Omega\times]0, T[)$ per ogni $T>0$, e u_0 e u_1 (dati iniziali) sono funzioni date in $L^2(\Omega)$. Il problema ammette una e una sola soluzione. Indicata con $e_k(t)$ la soluzione fondamentale dell'operatore $ay''(t)+by'(t)+(c-z_k)y(t)$ e con $\Lambda_k[\varphi]$ la distribuzione scalare $\int_0^\infty e_k(t)\varphi(t)dt$, vengono studiate le proprietà degli operatori di Green

$$G[\varphi]f=\sum_{k=1}^{\infty}\Lambda_k[\varphi](f, u_k)u_k$$
$$G[\varphi]*m=\sum_{k=1}^{\infty}u_k\int_0^\infty\Lambda_k[\varphi(t+\tau)](m(x, \tau), u_k(x))d\tau,$$

ottenendo per la soluzione del problema la espressione

$$u[\varphi]=G[\varphi](au_1+bu_0)+G\left[-\frac{\partial\varphi}{\partial t}\right]au_0+G[\varphi]*m.$$

E. Magenes (Genova)

1067:

Štraus, A. V. On spectral functions of an even order differential operator. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 67-70. (Russian)

This abstract continues the work of same Dokl. (N.S.) **111** (1956), 773–776 [MR **19**, 155].

M. M. Day (Urbana, Ill.)

1068:

Kostyučenko, A. G. On the spectral properties of elliptical operators. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 34–37. (Russian)

This paper discusses the relation between a condition on the kernel defining the resolvent of a self-conjugate differential operator and the boundedness of the eigenfunctions of the operator. *M. M. Day* (Urbana, Ill.)

1069:

Flekser, M. Š. Asymptotic behaviour of the spectral matrix of the theory-of-elasticity operator. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 470–472. (Russian)

In the papers of Levitan and Marchenko there are shown new methods for the examination of spectral characteristics of differential operators. In this note the author applies these methods to the problem of asymptotic behaviour of the spectral matrix elements of the theory-of-elasticity operator given in the finite region. The obtained formulas strengthen Pleijel's result [Ark. Mat. Astr. Fys. **27A** (1940), no. 13; MR **2**, 291] which was proved by Burčuladze [Soobšč. Akad. Nauk Gruzin. SSR **15** (1954), 193–200; MR **17**, 269] in another way.

Let Ω denote a finite region of Euclidean space E_3, and $\mathcal{L}$ the differential operator for the differential operation $-\Delta^*=-[b^2\Delta+(a^2-b^2)\text{grad div}]$ in Ω; $a>b>0$ are constants and Δ the Laplace operator. One considers the eigenvalue (μ) problem of this operator in Ω, $\mathcal{L}u-\mu^2u=0$, $u=\{u^i\}$, $i=1, 2, 3$. Let $\Theta(p, q, \mu)$ be the spectral matrix of the operator $\mathcal{L}$, with the elements $\vartheta_{ik}(p, q, \mu)=\sum u_n^i(p)u_n^k(q)$, where $u_n(p)$ and $u_n(q)$ are the eigenvectors and $p, q\in\Omega$; then one obtains for the elements ϑ_{ik} the following asymptotic formulas:

$$\overline{\lim_{\mu\to\infty}}\,\mu^{-2}\sup|\vartheta_{ik}-b^{-3}\delta_{ik}-(2\pi^2)^{-1}\partial^2 x_i x_k r^{-1}\,[\mathrm{Si}(\beta)-\mathrm{Si}(\alpha)]|$$
$$<10^8(a^{-3}+2b^{-3})/3\pi^2 h,$$

where $\vartheta=\vartheta(r, \mu)$ is the spectral function of Δ in all of E_3, δ_{ik} the Kronecker's symbol, Si(x) the integral-sinus, $r=|p-q|$, $\alpha=\mu r/a$, $\beta=\mu r/b$; h is an arbitrary positive number and Ω_h the set of the points in the region Ω with distance from the boundary not less than h; $p, q \in \Omega_h$.

D. P. Rašković (Belgrade)

1070:

Elazar, Silvio. Sull'integrazione in grande dei sistemi lineari ai differenziali totali in due variabili. Rend. Mat. e Appl. (5) **16** (1957), 221–249.

L'intégration globale d'un système linéaire aux différentielles totales à n fonctions inconnues de 2 variables indépendantes a été traitée dans le cas des domaines simplement connexes par Birindelli [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12** (1952), 518–523; MR **14**, 278] qui a d'ailleurs aussi abordé le cas des domaines multiplement connexes. [ibid. **14** (1953), 386–390; MR **15**, 129].

Le travail actuel de S. Elazar résout le problème pour un domaine d'ordre de connexion p quelconque. Le théorème général auquel il parvient fait intervenir le système homogène adjoint du système donné, et prend la forme d'un théorème d'alternative analogue à ceux que l'on rencontre dans la théorie des équations différentielles linéaires.

La question étant intimement liée aux propriétés topologiques du champ plan où doit se faire l'intégration, le mémoire débute par un rappel détaillé des propriétés des champs plans multiplement connexes, selon un exposé, en cours de publication, de L. de Vito.

M. Janet (Paris)

1071:

Mihaĭlov, V. On the reduction of a linear differential equation with constant coefficients to a system of equations of first order. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 6(78), 157–160. (Russian)

Der Verfasser beschäftigt sich mit der Frage, wann es möglich ist die Differentialgleichung mit konstanten Koeffizienten

$$(1) \qquad Lu \equiv \left(\sum_{\rho=0}^{N} \sum_{(i_n)=\rho} a_{i_0,\cdots,i_n} p_0^{i_0} \cdots p_n^{i_n}\right) u = 0$$

($a_{N,0,\cdots,0}=1$, $\partial/\partial x_i = p_i$) auf ein System erster Ordnung mit konstanten Koeffizienten zu überführen, dessen charakteristische Determinante gleich dem charakteristischen Polynom von (1) $\chi(\lambda, t_1, \cdots, t_n)=\sum \sum a_{i_0,\cdots,i_n} \lambda^{i_0} t_1^{i_1} \cdots t_n^{i_n}$ ist. Unter solcher Überführung versteht man dabei das folgende: 1. jede Komponente der Lösung des Systems ist eine Lösung von (1); 2. zu jeder Lösung von (1) existieren weitere Lösungen dieser Gleichung so, dass ein aus diesen Lösungen zusammengesetzter Vektor eine Lösung des Systems ist.

M. Zlámal (Brno)

1072:

Saltykow, N. Fonctions caractéristiques des équations aux dérivées partielles du second ordre. Bull. Acad. Serbe Sci. Cl. Sci. Math. Nat. (N.S.) **10** (1956), no. 2, 1–18.

"An application of Charpit's equation to the formation of ordinary differential equations corresponding to two partial differential equations of second order in two independent variables is considered. The notion of characteristic functions of the system in question is introduced. Their application to the formation of a complete integral of a system of two equations satisfying a Darboux-S. Lie involution is considered. A transformation of these latter conditions of involution is considered in order to obtain a second order partial differential equation in involution with a given one." (Author's summary.)

M. Steinberg (Los Angeles, Calif.)

1073:

Bellman, Richard; and Osborn, Howard. Dynamic programming and the variation of Green's functions. J. Math. Mech. **7** (1958), 81–85.

In this paper the functional equation technique of dynamic programming is applied to elliptic partial differential operators, in particular the Laplace operator, to determine the classical Hadamard variational formula for the corresponding Green's function. Let R be a bounded, connected region of Euclidean n-space with boundary ∂R of class C_2. Let Δu and u_ρ denote respectively the Laplacian of u and the limiting value of u on ∂R. Then for suitable functions v on R and w on ∂R the boundary value problem

$$(1) \qquad \Delta u = v, \quad u_\rho = w$$

has the unique solution

$$(2) \qquad u(p)=\int_{q\in R} g(p, q)v(q)+\int_{q\in\partial R} g_n(p, q)w(q),$$

where $g(p, q)$ is the suitably normalized Green's function for R and $g_n(p, q)$ is the exterior normal derivative of $g(p, q)$ on ∂R.

The principal result may be summarized as follows. Let φ be a non-negative function of class C_2 on ∂R and let R^* be the region bounded by the surface ∂R^* obtained from ∂R by a displacement $\delta n=\varepsilon\varphi$ along the interior normal. Let $g(\varepsilon, p, q)$ denote the Green's function for R^* and let $\delta g(p, q)=g(\varepsilon, p, q)-g(p, q)$. The Hadamard relation between δg and δn is shown to be

$$(3) \qquad \delta g(p, q)=\int_{s\in\partial R} \delta n(s) g_n(p, s) g_n(q, s)+o(\varepsilon).$$

The method of derivation of the variational formula for second order partial differential operators other than Laplace's operator is indicated.

P. E. Guenther (Cleveland, Ohio)

1074:

Lopatins'kiĭ, Ya. B. Boundary properties of solutions of second-order linear differential equations of elliptic type. Dopovidi Akad. Nauk Ukrain. RSR **1956**, 107–112. (Ukrainian. Russian summary)

1075:

Mihlin, S. G. Degenerate elliptic equations. Vestnik Leningrad. Univ. Ser. Mat. Fiz. Him. **9** (1954), no. 8, 19–48. (Russian)

1076:

Owens, O. G. A characterization of the everywhere regular solution of the reduced wave equation. Trans. Amer. Math. Soc. **88** (1958), 388–399.

Let $u(x, y)$ be a C^2 solution of $\Delta u+u=0$ in the entire plane which is such that $F(\theta)=\int_0^\infty u(r\cos\theta, r\sin\theta)dr$ converges uniformly in θ and satisfies a uniform Hölder condition with exponent $\alpha>\frac{1}{2}$. It is proved that $u(x, y)$ is determined uniquely by $F(\theta)$ and is given by

$$u(x, y)=\frac{1}{2\pi}\int_0^{2\pi} F(\phi)\cos[r\sin(\phi-\theta)]d\phi + \frac{1}{2\pi^2}\int_0^{2\pi} F(\phi)\int_0^{\pi/2} \frac{\Gamma(x, y, \tau, \phi)}{\sin\tau} d\tau d\phi,$$

where

$$\Gamma=\sin[x\sin(\tau+\phi)-y\cos(\tau+\phi)]+\sin[x\sin(\tau-\phi)+y\cos(\tau-\phi)].$$

The form of this representation is easily inferred by consideration of the Fourier series expansion $u(x, y)=\sum A_k(r)\cos k\theta+B_k(r)\sin k\theta$, and this approach yields a uniqueness theorem. The main task of the paper remains; this is to show that the function $u(x, y; F)$ defined from $F(\theta)$ by the integral formula does indeed satisfy $\int_0^\infty u(r\cos\theta, r\sin\theta; F)dr=F(\theta)$. The author suggests that the difficulties in the proof are related to the fact that, for a non-trivial C^2 solution of the differential equation, the convergence of $\int_0^\infty |u(r\sin\theta, r\cos\theta)|dr$ is not uniform.

R. McKelvey (Boulder, Colo.)

1077:

Yeh, G. C. K.; and Martinek, J. Disturbance of a many-dimensional field satisfying the Helmholtz equation due to the presence of a hyperplane boundary. Proc. Roy. Soc. London. Ser. A. **246** (1958), 423–428.

For an n-dimensional field satisfying the Helmholtz equation, the authors establish a theorem expressing the disturbance function, due to a hyperplane boundary, in terms of the undisturbed function. The field may have singularities on either or both sides of the boundary. Explicitly, they prove the following result.

Let $\phi_k(x_1)=\phi_k(x_1, \cdots, x_n)$, $k=0, 1$, be a real, single-valued, twice continuously differentiable solution of the Helmholtz equation $\Delta_n{}^2\phi_k+\lambda\phi_k=0$ with $\Delta_n{}^2=\sum_{i=1}^n \partial^2/\partial x_i{}^2$ and with $\lambda=$a real constant, having singularities only in $(-1)^k x_1>0$; then the functions $\phi_+(x_1)=\phi_+(x_1, \cdots, x_n)$ and $\phi_-(x_1)=\phi_-(x_1, \cdots, x_n)$, defined appropriately in terms of $\phi_0(x_1)$ and $\phi_1(x_1)$, satisfy the conditions (i) $\Delta_n{}^2(\phi_+-\phi_0)+\lambda(\phi_+-\phi_0)=0$ for $x_1>0$, (ii) $\Delta_n{}^2(\phi_--\phi_1)+\lambda(\phi_--\phi_1)=0$ for $x_1<0$, and (iii) $\phi_+=\mu\phi_-$ and $\nu_1\partial\phi_+/\partial x_1+\nu_3\phi_+=\nu_2\partial\phi_-/\partial x_1+\nu_4\phi_-$ as $x_1\to 0$, where μ and the ν_j are constants.

E. F. Beckenbach (Los Angeles, Calif.)

1078:

Levitan, B. M. Letter to the editor. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 599. (Russian)

An incorrect diagram on p. 355 of the author's article in Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 337–376 [MR **18**, 312], is here replaced by a correct one.

1079:

Bitsadze, A. V. On elliptical systems of second order partial differential equations. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 983–986. (Russian)

Let

$$Au_{xx}+2Bu_{xy}+Cu_{yy}+A_1u_x+B_1u_y+C_1u=0$$

be a system of second-order linear partial differential equations, with $u=(u_1, u_2, \cdots, u_n)$ and A, B, C, A_1, B_1, C_1 given as real $(n\times n)$ matrices continuous on a region of the xy-plane. Further, let the system be assumed strongly elliptic (in the sense that the quadratic form corresponding to the second-order terms is positive definite). If the coefficients are sufficiently smooth, then, for the Dirichlet problem on smoothly bounded simply-connected subregions, there is valid the Fredholm alternative: the inhomogeneous problem always has a solution, provided the corresponding homogeneous problem admits only the trivial solution. In the case of strongly elliptic equations of the form

$$(Au_x+Bu_y)_x+(Bu_x+Cu_y)_y+A_1u_x+B_1u_y+C_1u=0$$

with A_1 and B_1 symmetrical, a sufficient condition is that the matrix $A_1x+B_1y-2C_1$ be positive semi-definite over the subregion. Examples are included to show that the Dirichlet problem may or may not have a (unique) solution when the system fails to be strongly elliptic.

M. G. Arsove (Seattle, Wash.)

1080:

Teodorescu, N. Sur l'unicité de la forme de l'équation du conoïde caractéristique. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. Fiz. **7** (1956), no. 1–4, 7–14. (Romanian. Russian and French summaries)

The object of this note is to give a rigorous proof of a result due to Hadamard, that the square of the geodesic distance is the unique solution of $\Delta_1\Gamma=4\Gamma$, where Δ_1 is Beltrami's first differential parameter in a Riemannian space of metric ds^2. Actually the following more general result is proved: If the equation $\Delta_1 G=4\lambda^2 G$, where λ is a function of $(x_1, x_2, \cdots, x_n)$, has holomorphic coefficients, it has a unique holomorphic solution vanishing on the characteristic conoid with vertex O, and this solution is the square of the geodesic distance from O in the Riemannian space of metric $d\sigma^2=\lambda^2 ds^2$.

E. T. Copson (St. Andrews)

1081:

Drăgan, I. Observations sur la méthode de Darboux pour l'équation de Laplace $s+ap+bq+cz=0$. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **8** (1957), no. 2, 151–155. (Romanian. Russian and French summaries)

In this note, the author gives a new proof of Goursat's theorem which states that in the case of the partial differential equation of the title, Darboux's method is successful if and only if Laplace's method is successful. The proof is given by exhibiting at every step of Darboux's method a corresponding step of Laplace's method.

A. Erdélyi (Pasadena, Calif.)

1082:

Szmydt, Z. Sur l'existence de solutions de certains nouveaux problèmes pour un système d'équations différentielles hyperboliques du second ordre à deux variables indépendantes. Ann. Polon. Math. **4** (1957), 40–60.

Let R be the rectangle $|x|\leq a$, $|y|\leq b$; and let $y=\gamma(x)$ for $|x|\leq a$, $x=\lambda(y)$ for $|y|\leq b$ be two continuous arcs in R. For a vector $u=(u^1, \cdots, u^n)$, let $|u|=\max(|u^1|, \cdots, |u^n|)$. Let $F(x, y, u, p, q)$, $G(x, u, q)$, $H(y, u, p)$ be continuous vector functions defined for (x, y) in R and $|u|, |p|, |q|\leq r$. Let u_0 be a constant vector and $|u_0|, |G|, |H|\leq c<r$. The author is concerned with the question of the existence of functions $u(x, y)$ of class C^1 with continuous mixed derivatives u_{xy} satisfying $u_{xy}=F(x, y, u, u_x, u_y)$, $u(x_0, y_0)=u_0$, $u_x(x, y)=G(x, u(x, y), u_y(x, y))$ for $y=\gamma(x)$ and either

(I) $u_y(x, y)=H(y, u(x, y), u_x(x, y))$ for $x=\lambda(y)$, or

(II) $u(\lambda(y), y)=u_0+\int_{y_0}^{y} H(t, u(\lambda(t), t), u_x(\lambda(t), t))dt$.

It is shown that if F, G, H satisfy uniform Lipschitz conditions with respect to (p, q), q, p, respectively, with sufficiently small Lipschitz constants, then the problem (I) has a solution and if, in addition, $\lambda(y)$ is of class C^1, then problem (II) has a solution on R. Several variants of this result are given. For example, if in problem (I), G does not depend on q, or if $\gamma(x)\equiv$const, the Lipschitz condition on F can be replaced by a condition of the form $|F(x, y, u, p, q)-F(x, y, u, p^*, q^*)|\leq w_1(y, |p-p^*|, x)+w_2(x, |q-q^*|, y)$, where $w_i(s, v, t)$ are continuous, nondecreasing in v and the initial value problems $dv/ds=\pm w_i(s, v, t)$, $v(s_0)=0$ for fixed t, s_0 have the unique solution $v\equiv 0$. [See also Cili-

berto, Ricerche Mat. 4 (1955), 15–29; MR 17, 621.] The proofs depend on Schauder's fixed-point theorem.

P. Hartman (Baltimore, Md.)

1083:

Szmydt, Z. Sur l'existence d'une solution unique de certains problèmes pour un système d'équations différentielles hyperboliques du second ordre à deux variables indépendantes. Ann. Polon. Math. 4 (1958), 165–182.

The author gives several theorems asserting the uniqueness of the solutions for problems (I), (II) mentioned in the review above. One of his results implies the following: If $x=\lambda(y)$ is of class C^1 and if F, G, H satisfy uniform Lipschitz conditions with respect to (u, p, q), (u, q), (u, p), respectively, with sufficiently small Lipschitz constants, then (II) has a unique solution on R. The "sufficiently small" is specified in terms of the sides $2a$, $2b$ of R and of $w=\max|\lambda'(y)|$ for $|y|\leq b$. Examples are given showing that if the limitation on the Lipschitz constants is omitted, then no matter how small R is, the solution need not be unique.

P. Hartman (Baltimore, Md.)

1084

Žitomirskiĭ, Ya. I. Cauchy problem for a second degree parabolic equation with variable coefficients. Dokl. Akad. Nauk SSSR (N.S.) 116 (1957), 913–916. (Russian)

On considère un opérateur de la forme $L+\partial/\partial t$, où

$$L=-\sum_{i=1,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)+\sum_i b_i(x)\frac{\partial}{\partial x_i}+c(x),$$

avec $a_{ij}=a_{ji}$, $\sum a_{ij}(x)\xi_i\xi_j\geq\alpha^2\sum\xi_i^2$, $\alpha>0$, $c(x)>-c$, $c>0$. On étudie le problème de Cauchy, par des méthodes hilbertiennes; l'A. considère L dans un espace de fonctions de carré sommable avec poids $\exp(-g(x))$, $g(x)$ dépendant des a_{ij} et b_i — mais à l'aide d'une hypothèse supplémentaire fort restrictive.

J. L. Lions

1085:

Blyumenfel'd, V. N. On uniqueness of the limit distribution for a system of stochastic differential equations. Dokl. Akad. Nauk SSSR (N.S.) 111 (1956), 739–741. (Russian)

The author remarks that the (weakly) parabolic equation

$$\frac{\partial p}{\partial t}=\sum a_{ij}\frac{\partial^2 p}{\partial x_i\partial x_j}+\sum b_j\frac{\partial p}{\partial x_j}+cp\quad(i,j=1,2,\cdots,n),$$

where $\sum a_{ij}(x,t)\zeta^i\zeta^j\geq 0$ and $c(x,t)\leq h$ for all $(x,t)\in E^n\times[0,T]$, has unique solutions in the entire space $E^n\times[0,T]$ under the assumption that $\lim_{R\to\infty}\max_{\|x\|\geq R}|p(x,t)|=0$, and that $p(x,t)$ is continuous in t, uniformly with respect to x. He also remarks that this problem arises in a treatment of Kolmogorov's second equation.

A. N. Milgram (Minneapolis, Minn.)

1086:

Sestini, Giorgio. Sopra un teorema di unicità in problemi unidimensionali analoghi a quello di Stefan. Boll. Un. Mat. Ital. (3) 12 (1957), 516–519.

The author claims to have a uniqueness proof of a Stefan-like problem which is far simpler than one given by J. Douglas, Jr. [Proc. Amer. Math. Soc. 8 (1957), 402–408; MR 19, 1060]. The proof is incorrect.

C. R. DePrima (Pasadena, Calif.)

1087:

***Bizadse, W. A. Zum Problem der Gleichungen vom gemischten Typus.** Mathematische Forschungsberichte, V. VEB Deutscher Verlag der Wissenschaften, Berlin, 1957. 59 pp. DM 13.20.

This book is a translation into German of the original in Russian [Trudy Mat. Inst. Steklov. 41 (1953); MR 16, 43].

C. S. Morawetz (New York, N.Y.)

1088:

Zerner, Martin. Solutions intégralement semi-régulières de certaines équations aux dérivées partielles. C. R. Acad. Sci. Paris 246 (1958), 874–877.

L'auteur considère l'équation

$$Lu\equiv\frac{\partial^2 u}{\partial x\partial y}+A\frac{\partial u}{\partial x}+B\frac{\partial u}{\partial y}+Cu=0$$

où A, B, C sont des opérateurs différentiels en z à coefficients indéfiniments différentiables en (x, y, z), et il démontre que, dans chaque ouvert borné, toute solution u intégralement semi-régulière en z est de la forme $u=D_x^p u_{1,n}+D_y^p u_{2,n}$, où $u_{1,n}$ (resp. $u_{2,n}$) est une fonction continue ainsi que ses n premières dérivées par rapport à y (resp. x) et toutes leurs dérivées, par rapport à z, n étant arbitraire. L'auteur étudie aussi l'équation $Lu=v$, où v vérifie certaines conditions, et considère l'équation $Lu=0$ dans le cas où L ne contient que x et y.

J. Sebastião e Silva (Lisbonne)

1089:

Nicolescu, Miron. La structure des solutions des équations aux dérivées partielles du type elliptique ou du type parabolique. Magyar Tud. Akad. Mat. Kutató Int. Közl. 1 (1956), 465–479 (1957). (Hungarian and Russian summaries)

In this article, the author summarizes his researches on the polyharmonic equation $\Delta^p u(x, y)\equiv(\partial^2/\partial x^2+\partial^2/\partial y^2)^p u=0$ and the related polycaloric equation $(\partial^2/\partial x^2-\partial/\partial t)^p u(x,t)=0$. His exposition includes a discussion of the polyharmonic and polycaloric functions as well as a statement of some of the interesting questions that still remain unanswered. Complete details may be found in the author's "Les fonctions polyharmoniques" [Hermann, Paris, 1936] as well as in Comment. Math. Helv. 10 (1937/8), 3–17; Akad. R. P. Romîne. Stud. Cerc. Mat. 5 (1954), 243–332; MR 16, 709].

M. O. Reade (Ann Arbor, Mich.)

1090:

Wienholtz, Ernst. Halbbeschränkte partielle Differentialoperatoren zweiter Ordnung vom elliptischen Typus. Math. Ann. 135 (1958), 50–80.

The author considers the elliptic operator with real coefficients

$$Au=-\sum_{\mu,\nu=1}^{n}\frac{\partial}{\partial x_\nu}\left(a_{\mu\nu}(x)\frac{\partial u}{\partial x_\mu}\right)+2i\sum_{\nu=1}^{n}b_\nu(x)\frac{\partial u}{\partial x_\nu}+i\sum_{\nu=1}^{n}\frac{\partial b_\nu}{\partial x_\nu}u$$

over the set $\mathring{\mathfrak{S}}$ of C^∞ functions with compact support in R_n. The first part of the paper is concerned with finding sufficient conditions for $S=A+q(x)$ to be essentially self-adjoint in $\mathring{\mathfrak{S}}$ (i.e., for the closure of S in $L^2(R_n)$ to be self-adjoint). He first proves this assuming that $q(x)$ and the coefficients of A are regular and that (1) $(Su, u)\geq\gamma(u, u)$ for all $u\in\mathring{\mathfrak{S}}$, where $\gamma>0$ and the inner product is that of $L^2(R_n)$. He then considers the case when $q(x)$ is only known to be locally square integrable satisfying

$$(2)\qquad\int_{|x-y|\leq 1}|q(y)|^2\cdot|x-y|^{-n+4-\alpha}\leq M$$

for some $\alpha>0$, and A is uniformly elliptic in R_n. In this case he proves that S is essentially self-adjoint in $\mathring{\mathfrak{S}}$ under slightly more restrictive regularity conditions on the coefficients of A. He also discusses the situation when $q(x)$

satisfies (2) in a bounded region and is bounded outside. A section is devoted to the case when R_n is replaced by its subset $0 \leq l < |x| < m \leq \infty$.

In proving his theorems, the author makes use of the fact that a hermitian operator in Hilbert space which satisfies (1) and has a dense range is essentially self-adjoint. That A satisfies (1) is easily proved. That its range is dense in $L^2(R_n)$ is proved by means of the differentiability theory for weak solutions of elliptic equations [cf., e.g., L. Nirenberg, Comm. Pure Appl. Math. **8** (1955), 648–675; MR **17**, 742]. The result for S then follows from the fact that it is sufficiently "close" to A.

Previously similar results were obtained for the operator $-\Delta+q(x)$ by T. Kato [Trans. Amer. Math. Soc. **70** (1951), 195–211; MR **12**, 781] and F. Stummel [Math. Ann. **132** (1956), 150–176; MR **19**, 283].

The paper concludes with a study of growth properties of the eigenfunctions of $-\Delta u+q(x)u=\lambda u$, and they are used to derive sufficient conditions for a real number λ not to be a eigenvalue. *M. Schechter* (New York, N.Y.)

1091:

Višik, M. I.; and Ladyženskaya, O. A. Boundary value problems for partial differential equations and certain classes of operator equations. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 6(72), 41–97. (Russian)

This paper is one of a series of survey papers prepared for a conference on functional analysis, held in conjunction with the all-Union Congress of Soviet mathematicians which took place in the summer of 1956, and which have been printed in several issues of the Uspehi. Its subject is the application of linear space theory, combined with some analytical inequalities, to the formulation and solution of boundary value problems for linear elliptic equations as well as equations of propagation of the type of the heat equation and the wave equation. The paper is divided into two chapters of roughly the same length on each of the two general topics, which are not integrated into a common frame of reference. A comprehensive summary is given of some recent results in the theory of boundary value problems, with a preponderant, though not exclusive, emphasis upon the work of the authors and the results of the Russian school.

Chapter I begins with the formulation of a generalization of the well-known Friedrichs extension for a semi-bounded linear operator and, within the framework of this definition, describes results on the Dirichlet problem for strongly elliptic systems, higher boundary value problems, the Dirichlet problem for elliptic equations degenerating at the boundary, the method of orthogonal projection, and regularity properties of solutions. Chapter II begins with a description of the notion of weak solution for the mixed initial-boundary value problem for equations of the type $\partial^2 u/\partial t^2+L(t)u=f$ and $\partial u/\partial t+L(t)u=f$, with $(-1)^m L$ a strongly elliptic operator of order $2m$. Here, as in Chapter I, detailed attention is given to the special case $m=1$. Analogous initial value problems for operator differential equations are discussed and the proof of uniqueness of weak solutions is sketched. Existence proofs are sketched using two formulations of the projection method due respectively to each of the authors, as well as by continuation in a parameter. The Galerkin approximation to the solutions is discussed and results on differentiability properties of solutions are stated. The paper concludes with remarks on some unsolved problems in the domain of equations of propagation.

On the whole, this is the best organized and most coherent summary of the topics covered that is available at the present time. *F. Browder* (New Haven, Conn.)

1092:

***Višik, M. I.; and Ladyženskaya, O. A. Boundary value problems for partial differential equations and certain classes of operator equations.** American Mathematical Society Translations, Ser. 2, Vol. 10, pp. 223–281. American Mathematical Society, Providence, R. I., 1958. iv+409 pp. $6.60.

Translation of the paper reviewed above.

1093:

Campanato, Sergio. Sul problema di M. Picone relativo all'equilibrio di un corpo elastico incastrato. Ricerche Mat. **6** (1957), 125–149.

Il s'agit d'un problème d'élasticité, proposé par M. Picone [Second colloque sur les équations aux dérivées partielles, Bruxelles, 1954, Thone, Liège, 1955, pp. 9–11], i.e., d'un problème aux limites pour un système différentiel elliptique. L'existence et l'unicité de la solution faible de ce problème a été donnée par le rapporteur [Ann. Mat. Pura Appl. (4) **41** (1956), 201–219; MR **19**, 748]. L'auteur donne une interprétation "usuelle" des conditions aux limites, pour ce problème faible. Il utilise à cet effet une adaptation des méthodes de G. Stampacchia [Ricerche Mat. **5** (1956), 3–24; MR **18**, 579] relatives à un problème voisin, mais pour une équation aux dérivées partielles au lieu d'un système.

J. L. Lions (Nancy)

1094:

Simonov, M. I. D'Alembert's formula for a hyperbolic system of second order. Černivec. Derž. Univ. Nauk. Zap. Ser. Fiz.-Mat. **19** (1956), no. 4, 70–74. (Ukrainian. Russian summary)

1095:

***Панов, Д. Ю. [Panov, D. Yu.] Численное решение квазилинейных гиперболических систем дифференциальных уравнений в частных производных. [Numerical solution of quasilinear hyperbolic systems of partial differential equations.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1957. 216 pp. 8.40 rubles.

The system considered is of the form

$$a_{11}u_x+a_{12}v_x+b_{11}u_y+b_{12}v_y=e_1,$$
$$a_{21}u_x+a_{22}v_x+b_{21}u_y+b_{22}v_y=e_2,$$

with real characteristics, where $a_{11}, a_{12}, \cdots, e_2$ are functions of x, y, u and v. The author begins with a description of Euler's method for solving an ordinary differential equation and develops the method for the hyperbolic system as a generalization of this. The treatment is elementary and lucid, with illustrative examples carried out in detail. An appendix describes Markov-Hermite interpolation, and tables of auxiliary functions are given. There is a brief apology for the absence of theoretical error estimates. *A. S. Householder* (Oak Ridge, Tenn.)

1096:

Vostrecov, B. A. On the Cauchy problem for hyperbolic linear systems. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 5(77), 197–204. (Russian)

Consider the system

$$(1)\quad \frac{\partial^{n_i}u_i}{\partial t^{n_i}}=\sum_{j=1}^{m}\sum_{k_0,\cdots,k_n} A_{ij}^{k_0,\cdots,k_n}(t)\,\frac{\partial^{k_0+\cdots+k_n}u_j}{\partial t^{k_0}\partial x_1^{k_1}\cdots\partial x_n^{k_n}}$$

$(i=1, \cdots, m;\ 0 \leq k_0+\cdots+k_n \leq n_j;\ k_0<n_i)$, with initial

conditions $\partial^k u_i/\partial t^k|_{t=0}=\Phi_i^k(x_1, \cdots, x_n)$ $(k=0, 1, \cdots, n_i-1; i=1, \cdots, m)$. The author's treatment of this system rests upon the possibility of representing the Cauchy data in the form

$$\Phi_i^k(x)=\lim_{N\to\infty}\int\varphi_{iN}^k(x_1p_1+\cdots+x_np_n; p_1, \cdots, p_n)d\sigma,$$

the integration being over the unit sphere $\sum_1^n p_i^2=1$. Under certain conditions the sequence φ_{iN}^k in turn is given by

$$\varphi_{iN}^k(z; p_i)=\int_{D_0}\Phi_i^k(s_1, \cdots, s_n)\psi_n(p_1s_1+\cdots+p_ns_n-z)ds,$$

where D_0 is the base of the characteristic cone associated with (1), and the functions ψ_n are, within certain limitations, arbitrary. Let $z=\zeta_1(t, p_i, t_0)$ and $z=\zeta_2(t, p_i, t_0)$ be the solutions of the equations $dz/dt=-\mu_1(t, p_i)$ and $dz/dt=-\mu_2(t, p_i)$, where p_i stands for $p_1, \cdots, p_n$; $\mu_1(t, p_i)=\min[0, \lambda_1(t, p_i)]$, $\mu_2(t, p_i)=\max[0, \lambda_m(t, p_i)]$; and $\lambda_1(t, p_i)<\cdots<\lambda_m(t, p_i)$ are the eigenvalues of the canonical form of (1). If the coordinates of points in D_0 satisfy the inequality $\zeta_1(0, p_i, t_0)\leqq\sum_1^n x_ip_i\leqq\zeta_2(0, p_i, t_0)$, then the Cauchy problem is solved for u_i in the region for which $\zeta_1(t, p_i, t_0)\leqq\sum_1^n x_ip_i\leqq\zeta_2(t, p_i, t_0)$ $(0\leqq t\leqq t_0)$ for all p_i such that $\sum_1^n p_i^2=1$. The extension to the inhomogeneous system is considered briefly.

R. N. Goss (San Diego, Calif.)

1097:

Eidel'man, S. D. Some properties of parabolic systems of the form $L^qU=0$. Černivec. Derž. Univ. Nauk. Zap. Ser. Fiz.-Mat. **19** (1956), no. 4, 88–96. (Ukrainian. Russian summary)

1098:

Eidel'man, S. D. Uniqueness of the solution of the Fourier problem for parabolic systems. Černivec. Derž. Univ. Nauk. Zap. Ser. Fiz.-Mat. **19** (1956), no. 4, 83–87. (Ukrainian. Russian summary)

1099:

Kautský, Jaroslav. The solution of a quasilinear parabolic differential equation with an absolute member of a special type by the method of finite differences. Apl. Mat. **2** (1957), 327–341. (Czech. Russian and English summaries)

The author deals with the system of differential equations

$$\frac{\partial u}{\partial t}=g(x, t, u)\frac{\partial^2 u}{\partial x^2}+h(x, t, u)\frac{\partial u}{\partial x}+f(x, t, u, y(x, t)),$$

$$\frac{\partial y}{\partial t}=f(x, t, u, y(x, t)), \quad y(x, 0)=0.$$

He proves (under certain assumptions on the coefficients and on the initial and boundary conditions) the existence and uniqueness of the classical solution of this system in the rectangle $Q=[0,1]\times[0,T]$. T is a function of $|f(x, t, u, y)|$, $|\partial g/\partial u|$, $|\partial h/\partial u|$. The solution ${}_M u_m{}^n$ is obtained by the method of "finite differences". The convergence theorem for the solutions ${}_M u_m{}^n$ is based on Arzela's theorem and the estimates of the differences of the finite differences solutions.

M. Pinl (Cologne)

1100:

Alaci, V. Au sujet d'une classe d'équations aux dérivées partielles à coefficients constants. Acad. R. P. Romîne. Baza Cerc. Şti. Timişoara. Stud. Cerc. Şti. Ser. Şti. Tehn. **3** (1956), no. 1–2, 9–15. (Romanian. Russian and French summaries)

Using ideas of his previous paper [same Stud. Ser. I **2** (1955), 9–12; MR **18**, 401], the author indicates a method by which one can obtain solutions (depending on arbitrary constants and functions) of linear equations with partial derivatives, with constant coefficients, in $n+1$ independent variables. If

$$(*)\quad \sum_1^n\left(a_i\frac{\partial^3 u}{\partial x_i^3}+b_i\frac{\partial^2 u}{\partial x_i^2}+c_i\frac{\partial u}{\partial x_i}\right)=a\frac{\partial^3 u}{\partial t^3}+b\frac{\partial^2 u}{\partial t^2}+c\frac{\partial u}{\partial t}+du,$$

one postulates a solution of the form $u=e^{\alpha t}\varphi(v)$ where $v=\sum_{j=1}^n A_jx_j+\lambda t$ and A_j, α, λ, are $n+2$ arbitrary constants. Substitution in (*) leads to an ordinary, third order, linear differential equation for $\varphi(v)$. This will be satisfied by an arbitrary function $\varphi(v)$, provided that all its coefficients vanish. Thus one obtains a system S of four equations in the $n+2$ arbitrary constants. The system S is in general consistent for $n\geqq 2$ (if $n=1$, the coefficients of (*) have to satisfy two conditions in order to insure the consistency of S). Some equations with partial derivatives of higher order can also be handled by this method.

E. Grosswald (Philadelphia, Pa.)

1101:

Lebedev, V. I. The use of nets for the Sobolev type of equation. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 1166–1169. (Russian)

On considère dans un ouvert Ω de R^n (probablement borné) de frontière S, trois opérateurs différentiels du deuxième ordre: L_0, L_1, L_2; L_0 est uniformément elliptique auto-adjoint dans Ω; L_1 et L_2 sont des opérateurs différentiels du deuxième ordre; tous les coefficients sont fonctions bornées de $x\in\Omega$ et indépendantes de t. On cherche $u(x, t)$, $x\in\Omega$, $t\in[0, l]$, solution de

$$(1)\qquad Lu=\frac{\partial^2}{\partial t^2}L_0u+\frac{\partial}{\partial t}L_1u+L_2u=f(x, t),$$

avec $u(x, 0)$, $\partial u(x, 0)/\partial t$ donnés, u nul sur $S\times[0, l]$. Définition d'une solution faible. Existence et unicité d'une telle solution, en renvoyant à Višik [Mat. Sb. N.S. **39(81)** (1956), 51–148; MR **18**, 215]. {Rev.: Par composition de (1) par L_0^{-1}, on est ramené à une équation différentielle dans un espace de Hilbert, à coefficients opérateurs continus, d'où le résultat.} Le but de l'A. est de construire la solution par la méthode des différences finies. L'A. adapte dans le cas actuel les méthodes usuelles.

J. L. Lions (Nancy)

1102:

Roşculet, Marcel N. Equations aux dérivées partielles, linéaires, d'ordre quelconque, itérées. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **9** (1957), 79–105. (Romanian. Russian and French summaries)

Let the operator E be defined by $E=\sum_0^n\Delta_{n-k}(\partial^k/\partial y^k)$, where Δ_j is a linear operator of order j in $\partial/\partial x_1, \cdots, \partial/\partial x_s$ $(j=0, \cdots, n)$; and let $\delta E/\delta y=\sum_1^n k\Delta_{n-k}(\partial^{k-1}/\partial y^{k-1})$. Theorem: If $U_0, \cdots, U_p$ are functions of $s+1$ variables $x_1, \cdots, x_s$ and y, which have $n(p+1)$ derivatives in a domain D, and which are solutions of $E(U)=0$, then the function $\sum_0^p U_jy^j$ is a solution of $E^{p+1}(U)=0$ in D. If this series representation is unique, then $[\delta E/\delta y]^p(U)=0$ and $E(U)$ must be of the form $\partial U/\partial y+E^*(\partial/\partial x_1, \cdots, \partial/\partial x_s)$. Two initial value problems for this operator are discussed. The results are generalized to cases where the operator $\partial/\partial y$ in $\partial/\partial y+E^*$ is replaced by a linear operator with variable coefficients.

N. D. Kazarinoff (Ann Arbor, Mich.)

1103:

Borok, V. M. Reduction to a single equation of an evolutional system of linear partial differential equations with constant coefficients. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 682–685. (Russian)

It is shown that any system of differential equations (1) $D_t u = P(D_x)u$ is equivalent to a set of equations $D_t^N v + \sum_{k>0} D_t^{N-k} P_k(D_x)v = 0$. Further, if (1) is hyperbolic, then order $P_k \leqq k$, so that (1) is equivalent to a system with order $P=1$. *L. Gårding* (Lund)

1104:

Goldhagen, E. Sur l'intégration explicite des systèmes de n équations aux dérivées partielles quasilinéaires du premier ordre à $n+1$ fonctions inconnues de deux variables indépendantes. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **7** (1956), no. 1, 51–70; **8** (1957), no. 1, 75–106. (Romanian. Russian and French summaries)

Continuing earlier work [Acad. R. P. Romîne. Bul. Şti. Sect. Şti. Mat. Fiz. **7** (1955), 623–644; MR **17**, 490] the author considers a system of n quasilinear partial differential equations in two independent variables x and y, and for $n+1$ unknown functions $z_1, \cdots, z_{n+1}$, assuming that the general solution is of the form $z_i = f_i(x, y, \phi, \phi_x, \phi_y, \phi_{xx}, \cdots)$, $i=1, \cdots, n+1$, where the f_i are given functions of x, y, of an "arbitrary" function ϕ and of a finite number of partial derivatives of ϕ, say up to and including order t.

In the first paper he assumes that there is no relation of the form

$$(*) \qquad F(x, y, z_1, \cdots, z_{n+1}, \phi, \phi_x, \cdots) = 0$$

satisfied identically in x and y, and involving partial derivatives of ϕ up to order $t-1$ at most. Under this condition he shows that the equations can be transformed to a canonical form, determines the form of the solution, and shows that for $n=2$ the solution so obtained is the general solution.

In the second paper the condition (*) is removed, and the corresponding results are obtained in this more general case. *A. Erdélyi* (Pasadena, Calif.)

1105:

Aleksandrov, A. D. Some theorems on partial differential equations of second order. Vestnik Leningrad. Univ. Ser. Mat. Fiz. Him. **9** (1954), no. 8, 3–17. (Russian)

1106:

Vallander, S. V. Non-linear partial differential equations of second order in two independent variables which are reducible to linear equations. Vestnik Leningrad. Univ. Ser. Mat. Fiz. Him. **9** (1954), no. 5, 19–34. (Russian)

DIFFERENTIAL ALGEBRA

1107:

***Kähler, Erich. Algebra und Differentialrechnung.** Mathematische Monographien, 1. Sonderdruck aus "Bericht über die Mathematiker-Tagung in Berlin vom 14. bis 18. Januar 1953", pp. 58–163. VEB Deutscher Verlag der Wissenschaften, Berlin, 1958. DM 9.00.

A reprinting under separate cover, but with the original pagination, of the article reviewed in MR **16**, 563.

1108a:

Dahmen, G. Zur Theorie der Differentialpolynome I. Differentialpolynome in einer Unbestimmten. Ann. Univ. Sarav. **6** (1957), 311–323.

1108b:

Dahmen, G. Zur Theorie der Differentialpolynome II. Differentialpolynome in mehreren Unbestimmten. Ann. Univ. Sarav. **6** (1957), 324–336.

The basic results of Ritt and of Raudenbush are proved anew. The results concern prime and perfect differential ideals in a differential polynomial ring $F\{y_1, \cdots, y_n\}$ over a differential field F (which is always assumed to be ordinary and of characteristic 0), and extensions of such an F. *E. R. Kolchin* (New York, N.Y.)

1109:

Kolchin, Ellis; and Lang, Serge. Algebraic groups and the Galois theory of differential fields. Amer. J. Math. **80** (1958), 103–110.

The main result of this paper is the following theorem.

Let F be a differential field of characteristic 0 with algebraically closed field of constants C, and let E be a strongly normal extension of F, all lying in a universal extension F^* with field of constants C^*. Let $\mathfrak{G}$ be the Galois group of E over F. Then there exists an algebraic group G defined over C, and a principal homogeneous space V for G, defined over the algebraic closure F_0 of F in E, having the following properties.

(a) V is a model of E over F; that is, any component of V has a point v generic over F_0 such that $E=F_0(v)$, and these components are all conjugates to each other over F.

(b) For each $\sigma \in \mathfrak{G}$ there exists a unique point $x_\sigma \in G_{C^*}$ (the group of points of G which are rational over C^*) such that $\sigma v = v x_\sigma$, and the mapping $\sigma \to x_\sigma$ ($\sigma \in \mathfrak{G}$) is a group isomorphism of $\mathfrak{G}$ onto G_{C^*}.

(c) This isomorphism is rational in the sense that $C(\sigma)=C(x_\sigma)$ and that σ' is a specialization of σ if and only if $x_{\sigma'}$ is a specialization of x_σ over C. Here the symbol $C(\sigma)$ denotes the field of constants of the compositum $E \cdot \sigma E$. *H. Levi* (New York, N.Y.)

POTENTIAL THEORY

See also 965, 970, 1089, 1338.

1110:

Tolsted, Elmer. Non-tangential limits of subharmonic functions. Proc. London Math. Soc. (3) **7** (1957), 321–333.

On sait que si u est sousharmonique dans le cercle-unité, avec moyenne de $|u|$ bornée sur les circonférences concentriques, il y a presque partout à la frontière une limite radiale (finie); lorsque u est harmonique, il y a même une limite "non-tangentielle" (c'est-à-dire, dans un secteur de demi-angle $<\pi/2$) mais ce n'est pas vrai en général. L'A., qui a déjà insisté sur ce point [Proc. Amer. Math. Soc. **1** (1950), 636–647; MR **12**, 609], montre ici qu'il y a une telle limite (p.p.) pour la moyenne dans un cercle de centre P_0 et rayon κs_0 ($0<\kappa<1$, $s_0=1-OP_0$) quand P_0 tend vers un point-frontière. Il y a même une telle limite pour u elle-même lorsque dans la représentation potentielle locale de u, il y a une densité f des masses satisfaisant à $\int (1-OP) f(P) \log^+ f(P) dP$ fini. On se

ramène à examiner le cas d'un potentiel de Green pour lequel les limites précédentes sont nulles.

M. Brelot (Paris)

1111:

Nikol'skiĭ, S. M. On a variation problem of Hilbert. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 573–575. (Russian)

The problem of Hilbert in question is the Dirichlet problem in the plane with prescribed discontinuity across a given closed curve [Math. Ann. **59** (1904), 161–186]. This is generalized as follows. Let g be a region in real n-space R_n bounded by a twice continuously differentiable surface Λ. For $Q \in \Lambda$ let $f_+(Q)$, $f_-(Q)$ denote the limit wherever it exists of $f(P)$ as $P \to Q$ along the interior, exterior normal to Λ at Q. Let $a(Q) \neq 0$, $b(Q)$, and $\kappa(Q)$ be given on Λ, where $a(Q)$ and $b(Q)$ are once continuously differentiable and $\kappa = f_+$ for some f with finite Dirichlet integral over g. Then the Dirichlet integral has a minimum subject to the boundary condition $af_+ - bf_- = \kappa$ on Λ and to the normalization $\int_\gamma f dy = 0$, where γ is a given $(n-1)$-parallelopiped in $R_n - \Lambda$.

W. H. Fleming (Providence, R.I.)

1112:

Gehring, F. W. The Fatou theorem for functions harmonic in a half-space. Proc. London Math. Soc. (3) **8** (1958), 149–160.

L'A. estende alle funzioni armoniche nel semispazio il teorema di Fatou e il suo inverso, dovuto a Loomis. Sia μ una misura definita in tutti gli insiemi di Borel del piano (ξ, η) e tale che $\iint_{-\infty}^{+\infty} |d\mu(\xi, \eta)|/(\xi^2+\eta^2+1)^{3/2} < \infty$. Diciamo che $D^*\mu(0, 0) = A$, se la funzione $(\pi t^2)^{-1} \int_\alpha^\beta \int_0^t d\mu(\rho, \theta)$ tende a $(2\pi)^{-1}(\beta-\alpha)A$ al tendere di t a $0+$, per ogni coppia $\alpha < \beta$, rimanendo inoltre limitata per $\alpha < \beta \leq \alpha + 2\pi$ e t sufficientemente piccolo; ovviamente $D^*\mu(0, 0) = A$ implica che $D\mu(0, 0) = A$, $D\mu$ essendo la derivata di μ nel senso abituale (caso $\alpha = 0$, $\beta = 2\pi$).

Si consideri allora la funzione $u(x, y, z)$ armonica nel semispazio R^+ delle z positive

$$\text{(I)} \quad u(x, y, z) = kz + \frac{z}{2\pi} \iint_{-\infty}^{+\infty} \frac{d\mu(\xi, \eta)}{[(x-\xi)^2+(y-\eta)^2+z^2]^{3/2}}$$

con k costante. L'A. dimostra: 1) Se $D^*\mu(0, 0) = A$, allora $u(x, y, z) \to A$ quando $(x, y, z) \to (0, 0, 0)$ su ogni raggio di R^+ uscente da $(0, 0, 0)$; se è solamente $D\mu(0, 0) = A$, allora $u(0, 0, z) \to A$ per $z \to 0+$ [quest'ultimo risultato é già noto, si veda ad es. G. C. Evans ed E. R. C. Miles, Amer. J. Math. **53** (1931), 493–516]. 2) Supposto in (I) k e μ non negative, se $u(x, y, z) \to A$ quando $(x, y, z) \to (0, 0, 0)$ su ogni raggio di R^+ uscente da $(0, 0, 0)$, allora $D^*\mu(0, 0) = A$; se $u(0, 0, z) \to A$ per $z \to 0+$, allora $D\mu(0, 0) = A$.

E. Magenes (Genova)

1113:

Vašarin, A. A. The boundary properties of functions having a finite Dirichlet integral with a weight. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 742–744. (Russian)

$W_{2(\omega)}^k$ is the class of functions f defined in a given plane domain Ω with boundary Γ such that

$$\iint_\Omega \Big[\sum_{m+n=k} \frac{k!}{m!n!} \Big(\frac{\partial^k f}{\partial x^m \partial y^n} \Big)^2 \Big] \sigma^\alpha dx dy < \infty,$$

where $0 \leq \alpha < 1$ and σ is a properly chosen positive function in Ω. Under suitable hypotheses the author shows that the sequence of the normal derivatives of f up to the $(k-1)$st have on Γ certain properties, and that conversely these properties of a sequence of functions on Γ imply the existence of an f in $W_{2(\omega)}^k$ with the given sequence as normal derivative sequence.

M. M. Day (Urbana, Ill.)

1114:

Zaharov, V. K. Imbedding theorems for a space having its metric degenerating at a finite number of internal points within a bounded domain. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 938–941. (Russian)

[See also # 1207.] Let D' be a domain of the plane bounded by a simple closed curve Γ, and the point $(0,0)$ interiorto Γ, and let Ω_0 be the set of piece wise smooth functions in D' which vanish on a neighborhood of Γ and of $(0, 0)$. A metric is introduced in Ω_0 in terms of the integral of a bilinear form in the gradients of a pair of elements of Ω_0, and Ω_0 is completed in this metric to a space Ω. The paper lists assorted conditions on the coefficient functions in the bilinear form and corresponding results on the behavior of all the functions on $D' \cup \Gamma$ which correspond to elements of Ω.

M. M. Day (Urbana, Ill.)

1115:

Malferrari, Angelo. Su alcune proprietà dei potenziali generalizzati di doppio strato obliquo. Ricerche Mat. **6** (1957), 151–161.

Vengono dimostrati due teoremi sul potenziale generalizzato di doppio strato "obliquo" relativo ad un operatore lineare ellittico del secondo ordine Mu, che generalizzano risultati noti [v. ad es. C. Miranda, Equazioni alle derivate parziali di tipo ellittico, Springer, Berlin, 1955; MR **19**, 421] nel caso degli ordinari potenziali di doppio strato. Precisamente siano $L(x, y)$ una opportuna funzione di Levi per l'operatore Mu (ad es. una soluzione fondamentale), Γ la frontiera del dominio Ω dello spazio euclideo ad n dimensioni in cui sono definiti i coefficienti di Mu, $l(x)$, per ogni x di Γ, un asse obliquo penetrante in Ω, $\lambda(x)$ il cosidetto asse coriflesso di $l(x)$ rispetto ad Mu e $\nu(x)$ l'asse conormale rispetto ad Mu. Si consideri il potenziale di doppio strato obliquo

$$w(x) = \int_\Gamma z(y) QL(x, y) d\sigma_y,$$

essendo

$$QL(x, y) = a_\lambda(y) \frac{dL(x, y)}{d\lambda(y)} + b(y) L(x, y)$$

con $a_\lambda(y)$ e $b(y)$ opportune funzioni. In ipotesi assai larghe sui coefficienti di Mu, su $L(x, y)$ e su Γ, viene dimostrata la validità della formula, per quasi-tutti gli x_0 di Γ se z è sommabile su Γ, e per ogni x_0 di Γ se z è continua su Γ:

$$a_l(x_0) \frac{dw(x)}{dl(x_0)} + \beta(x_0) w(x) - a_l(x_0) \frac{dw(x')}{dl(x_0)} - \beta(x_0) w(x') \to 0,$$

dove $x \to x_0$ stando su $\nu(x_0)$, x' è il simmetrico di x rispetto a x_0, e $a_l(x)$ e $\beta(x)$ sono opportune funzioni legate a $a_\lambda(x)$ e $b(x)$; viene anche dimostrata la possibilità di prolungare $w(x)$ in $\Omega + \Gamma$, se $z \in C^{1,h}(\Gamma)$, in modo che $W(x) \in C^{1,h}(\Omega + \Gamma)$.

E. Magenes (Genova)

1116:

Inoue, Masao. Positively infinite singularities of solutions of linear elliptic partial differential equations. J. Inst. Polytech. Osaka City Univ. Ser. A. **8** (1957), 43–50.

Considerata nell'insieme Ω aperto e connesso del piano (x, y) l'equazione differenziale

$$\text{(1)} \quad \mathcal{F}u = \Delta u + au_x + bu_y + \varphi(x, y, u) = 0$$

(Δu laplaciano), l'A. dà alcune interessanti condizioni perchè un insieme di $\mathcal{F}$-capacità nulla (un insieme chiuso $E \subset \Omega$ è tale se esistono un insieme aperto $O \supset E$ e una soluzione u della (1) continua in $O - E$ con le derivate che

compaiono in $\mathscr{F}u$, tale che $\lim_{(x,y)\to(\xi,\eta)} u(x, y)=+\infty$ per ogni (ξ, η) della frontiera di $O-E$ appartenente a E; si scrive allora $C_{\mathscr{F}}(E)=0$) sia anche di Δ-capacità (capacità logaritmica) nulla. In particolare si ha che $C_{\mathscr{F}}(E)=0$ è equivalente a $C_{\Delta}(E)=0$ se

$$\mathscr{F}u=\Delta u+au_x+bu_y+f \tag{2}$$

con a, b, f funzioni differenziabili con continuità in Ω. Come conseguenza l'A. estende all'operatore (2) noti risultati sulla eliminazione delle singolarità delle funzioni armoniche. *E. Magenes* (Genova)

1117:

Ohtsuka, Makoto. Sur un théorème de M. Kishi. Proc. Japan Acad. **32** (1956), 722–725.

Let Ω be a locally compact space and let $\Phi(P, Q)>-\infty$ be a real-valued function defined on $\Omega\times\Omega$. If μ is a finite, positive measure with compact support, then one has the potential $U^{\mu}(P)=\int \Phi(P, Q)d\mu(Q)$. The author says the "principle of the maximum in the sense of Ugaheri generalized by Kishi" (pr. max. gen. Kishi) holds if and only if for each compact set $K\subset\Omega$ there exists a constant $c=c(\Omega, \Phi, K)$ such that $U^{\mu}(P)\leq c\cdot\sup_{Q\in K} U^{\mu}(Q)$ holds throughout Ω for each μ carried by K. The author says that the "principle of upper boundedness" (pr. limit sup.) holds if and only if the existence of a finite upper bound for $U^{\mu}(P)$ on the support of μ implies the existence of a finite upper bound for $U^{\mu}(P)$ throughout Ω. The author says that the "principle of continuity" (pr. cont.) holds if and only if the finiteness and continuity of $U^{\mu}(P)$ on the support of μ implies the same properties of $U^{\mu}(P)$ throughout Ω. The author introduces the "weak principle of the maximum of Ugaheri" (pr. max. Ugaheri faible) as follows: for each μ carried by a compact K there must exist a constant $c=c(\Omega, \Phi, K)$ such that $U^{\mu}(P)\leq c\cdot\sup_{Q\in S_{\mu}} U^{\mu}(Q)$ holds throughout Ω. Then the author shows that the following schemes hold: (i) pr. max. Ugaheri faible $\rightleftharpoons$ pr. lim. sup. $\rightleftharpoons$ pr. cont.; (ii) pr. lim. sup. $\rightleftharpoons$ pr. max. gen. Kishi, for $\Phi>0$.

M. O. Reade (Ann Arbor, Mich.)

1118

Ohtsuka, Makoto. Les relations entre certains principes en théorie du potentiel. Proc. Japan Acad. **33** (1957), 37–40.

Divers principes de "régularité" (principes du maximum de Frostman et de Ugaheri, principe de continuité de Evans-Vasilesco, etc.) en théorie générale du potentiel ont fait l'objet de travaux récents: voir notamment G. Choquet [C. R. Acad. Sci. Paris **243** (1956), 635–638; MR **18**, 295]; M. Kishi [Proc. Japan. Acad. **32** (1956), 314–319; MR **18**, 29]; M. Ohtsuka [≠ 1117 ci-dessus]; N. Ninomiya [J. Inst. Polytech. Osaka City Univ. Ser. A. **8** (1957), 51–56; MR **19**, 262]. L'auteur fait le point des résultats obtenus dans cette voie et ajoute quelques remarques lui permettant de former le schéma complet des relations logiques entre 9 de ces principes.

J. Deny (Strasbourg)

1119:

Choquet, G.; et Deny, J. Ensembles semi-réticulés et ensembles réticulés de fonctions continues. J. Math. Pures Appl. (9) **36** (1957), 179–189.

Let X be a topological space, F a family of couples (x, σ), where x is a point of X and σ is a positive Radon measure on X having compact support and vanishing on the point x, and G a family of positive Radon measures on X with compact supports. Let E be the set of all continuous real-valued functions u on X such that $\int u d\sigma\leq u(x)$ for every (x, σ) in F and $\int u d\tau\leq 0$ for every τ in G. Then E is a convex cone which is closed in the topology of uniform convergence on compact sets, and also is closed under taking the minimum of two functions (i.e., is a lower semi-lattice). The authors' principal result is the converse theorem that any such cone of functions can be characterized by a set of inequalities of the above type. If the cone E is compact around its vertex (i.e., if the functions of E, suitably normalized, form an equicontinuous family) then the compact supports of the above measures can be restricted to finite sets, and the above theorem becomes a direct generalization of the classical Harnack inequalities for positive harmonic functions. *L. H. Loomis* (Cambridge, Mass.)

1120:

Elianu, I. P. Sur les formes différentielles polyharmoniques. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **9** (1957), 233–240. (Romanian. Russian and French summaries)

"L'Auteur définit, sur un espace de Riemann, les formes différentielles fermées (cofermées), homologues (cohomologues) à zéro d'ordre supérieur, et les formes polyharmoniques. Dans le cas des espaces compacts et sous certaines hypothèses de dérivabilité, on démontre que ces formes coincident respectivement avec les formes fermées (cofermées), homologues (cohomologues) a zéro proprement dits, ou avec les formes harmoniques. Dans le cas des espaces euclidiens, on donne un développement du type d'Almansi pour les formes polyharmoniques, et dans le cas des espaces analytiques, on montre que les formes polyharmoniques sont analytiques. Finalement, on met en evidence une classe de fonctions complexes non monogènes qui contiennent la classe des polynômes aréolaires." (Résumé de l'auteur) *D. Pedoe* (Khartoum)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 838, 1073, 1131.

1121:

***Gelfond, A. O. Differenzenrechnung.** Hochschulbücher für Mathematik, Bd. 41. VEB Deutscher Verlag der Wissenschaften, Berlin, 1958. viii+336 pp. DM 40.00.

A translation, by Helmut Limberg and Gerhard Tesch, of "Isčislenie konečnyh raznosteĭ", Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952 [MR **14**, 759.]

1122:

Simonsen, W. On divided differences in terms of derivatives. Skand. Aktuarietidskr. **40** (1957), 26–45.

The present paper considers the general problem of expanding a divided difference of a function in terms of its successive derivatives at specified points, and some special cases. Formulas are derived for: (1) central differences of even orders at the point a in terms of derivatives of even orders at a; (2) means of central differences of odd orders at a in terms of derivatives of odd orders at a; (3) central differences of odd orders at $a+h/2$ in terms of derivatives of odd orders at $a+h/2$; (4) means of central differences of even orders at $a+h/2$ in terms of derivatives of even orders at $a+h/2$.

D. Moskovitz (Pittsburgh, Pa.)

1123:

Kemperman, J. H. B. A general functional equation. Trans. Amer. Math. Soc. **86** (1957), 28–56.

For a large number of classes Ω of real functions of a real variable it was shown by the reviewer [Nieuw Arch. Wisk. (2) **23** (1951), 194–218; Nederl. Akad. Wetensch. Proc. Ser. A. **55** (1952), 145–151; MR **13**, 332, 830] that if $\Delta_h f \in \Omega$ for each h, then f has the form $g+H$, where $g \in \Omega$, and H is an additive function ($\Delta_h f$ is defined by $(\Delta_h f)(x)=f(x+h)-f(x)$). This property of Ω is called the difference property. Classes Ω having the difference property are, e.g., the classes of all continuous functions, all differentiable functions, and all Riemann integrable functions. The class of all measurable functions and the class of all bounded functions do not have the difference property.

It is well known that an additive function is linear whenever it is bounded on some set of positive measure, so that in many cases it can be shown that if $\Delta_h f \in \Omega$ for each h, and if f is bounded on some set of positive measure, then $f \in \Omega$. This property is called the weak difference property.

The author generalizes these results in several directions. (i) Instead of functions of a single variable, he considers functions in m-dimensional euclidean space E_m, so that both x and h are replaced by vectors. (ii) The functions f need to be defined only in some open subset of E_m. (iii) In some cases, restrictions are made on h. (iv) Instead of one function f, the author considers several functions.

A typical result (in a simplified form) is as follows. Let Ω be a class such that f, $g \in \Omega$ imply $f-g \in \Omega$, and $f(x+h) \in \Omega$ for all h, and assume either that all $f \in \Omega$ are measurable or that all $f \in \Omega$ are bounded in every bounded interval. Let $b_0, \cdots, b_k$ be different constants, and let $f_0, \cdots, f_k$ be real functions defined on E_m. For each h, $\varphi_h(x)$ is defined as $\sum_{i=0}^{k} f_i(x+b_i h)$. Finally, assume that Ω has the weak difference property, that f_0 is bounded on some set of positive measure, and that $\varphi_h \in \Omega$ for each h. Then we have $f_0 \in \Omega$.

The author also gives a sufficient condition (of a general type) for a class Ω to have the difference property (Theorem 7.1), with application to several new cases.

The paper ends with an application to a new proof of Skitovič's sufficient condition for normal distribution of a number of independent random variables [Izv. Akad. Nauk SSSR. Ser. Mat. **18** (1954), 185–200; MR **16**, 52].

N. G. de Bruijn (Amsterdam)

1124:

***Boole, George. Calculus of finite differences. Edited by J. F. Moulton. 4th ed.** Chelsea Publishing Company, New York, 1957. xii+336 pp. $4.95.

1125:

Fort, Tomlinson. The five-point difference equation with periodic coefficients. Pacific J. Math. **7** (1957), 1341–1350.

The author studies the partial difference equation

$$(1)\quad k_1(i,j)y(i-1,j)+k_2(i,j)y(i+1,j)+k_3(i,j)y(i,j-1)+k_4(i,j)y(i,j+1)+k_5(i,j)y(i,j)=0,$$

where $k_s(i,j)$ is defined for $i=1, 2, \cdots, n\omega-1$ and $j=1, 2, \cdots, \omega-1$, with $n>1$, $\omega>1$, and $k_s(i+\omega,j)=k_s(i,j)$. To facilitate a search for a solution of (1) satisfying $y(i+\omega,j)=\rho y(i,j)$ for some nonzero constant ρ, a special choice of "fundamental domain" [cf. Fort, Amer. Math. Monthly **64** (1957), 161–167; MR **19**, 41] is made. After this the methods follow lines traditional, e.g., in the theory of ordinary differential equations with periodic coefficients. A characteristic equation is found for ρ. If all its roots $\rho_1, \cdots, \rho_L$ are simple, then there exists a distinguished fundamental system $y_1, \cdots, y_L$ of solutions of (1) such that $y_l(i+\omega,j)=\rho_l y_l(i,j)$. If multiple roots occur the distinguished fundamental system exhibits an expected pattern of complications. {There are numerous minor errors in both text and bibliographical reference.}

W. Strodt (New York, N.Y.)

1126:

Koval', P. I. Reducible systems of difference equations and stability of their solutions. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 6(78), 143–146. (Russian)

The finite system of linear difference equations $x_{s+1}=A_s x_s+a_s$ ($s=0, 1, 2, \cdots$) is considered, where $\{x_s\}$ is a sequence of unknown vectors, A_s a given bounded sequence of non-singular matrices and $\{a_s\}$ a given bounded sequence of vectors. This system is called reducible if there exist transformations T_s (which together with their inverses are bounded by some constant not depending on s) for which the matrices B_s in the system $y_{s+1}=B_s y_s+b_s$ ($s=0, 1, 2, \cdots$) obtained by substituting $x_s=T_s y_s$ into the previous equations, are constant. It is shown that $x_{s+1}=A_s x_s+a_s$ is reducible if its coefficients are periodical.

A system is almost periodical if there exist transformations T_s which together with their inverses are bounded and for which the matrix $B_s=T_{s+1}^{-1}A_s T_s$ has a limit when $s \to \infty$. It is shown that if the A_s are the sums of two matrices, one periodical and the other having the null matrix as a limit, then the system is almost periodical. The author discusses, finally, the stability of the solution of the system of difference equations in the sense of Liapunov and Perron [Perron, J. Reine Angew. Math. **161** (1929), 41–64] and shows that in the case of reducible or almost reducible systems it is sufficient to discuss the stability of the reduced system of difference equations (i.e., the one with constant coefficients). The system with periodical coefficients ($A_{k+s}=A_s$) is asymptotically stable if all characteristic values of the matrix $P=A_{k-1}A_{k-2}\cdots A_0$ are smaller than 1 in absolute value.

U. W. Hochstrasser (Lawrence, Kans.)

1127:

Ghermănescu, Mihail. Sur l'équation fonctionnelle de Cauchy. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1(49)** (1957), 33–46.

The author gives two new methods for determining the continuous or measurable solutions of $f(x+y)=f(x)+f(y)$. He also solves a number of related equations, such as

$$\frac{x_1 f(x_2)-x_2 f(x_1)}{x_1-x_2}=f\left(\frac{x_1+x_2}{2}\right)-\frac{x_1+x_2}{2}f'\left(\frac{x_1+x_2}{2}\right),$$

$\phi(\alpha x)=\phi(x)$, $f(\alpha x+\beta y)-af(x)=\phi(x,y)$, $f(\alpha x+\beta y)-af(x)-bf(y)=u(x,y)$.

R. P. Boas, Jr. (Evanston, Ill.)

1128:

Domb, C.; and Fisher, M. E. On iterative processes and functional equations. Proc. Cambridge Philos. Soc. **52** (1956), 652–662.

1129:

Ghermanescu, Michel. Une classe d'équations fonctionnelles linéaires. C. R. Acad. Sci. Paris **245** (1957) 274–276.

This note considers the linear functional equation (1) $f(M)+\sum_{k=1}^{n} a_k(M)f^k(M)=0$, in which $f(M)$ is the unknown function of a point M in a given domain D of

a given multi-dimensional space, $f^k(M)$ is the kth iterate of $f(M)$ with respect to the given argument function $\theta(M)$, and each coefficient $a_k(M)$ is assumed to satisfy $a_k(\theta(M))=a_k(M)$. The author associates two equations with (1). The first is the characteristic equation, a polynomial in $\lambda(M)$: (2) $\lambda^n(M)+\sum_{k=1}^{n} a_k(M)\lambda^{n-k}(M)=0$, of which the roots $\lambda_k(M)$ also satisfy $\lambda_k(\theta(M))=\lambda_k(M)$. The second is the determinantal equation (3) $|f^{i+j-2}(M)|=0$, $i, j=1, \cdots, n+1$. He then states a number of theorems concerning the existence of solutions of (1) and (3) and the structures of the associated manifolds of solutions. Two of them are the following: (I) The necessary and sufficient condition that (1) admit solutions not identically zero in D is that at least one of the functional equations (4) $f^1(M)-\lambda_k(M)f(M)=0$, $k=1, \cdots, n$, admit such solutions. (IV) The integration of the functional equation (3) reduces to the integration of an equation of type (1) in which the coefficients $a_k(M)$ are chosen so that the equations (4) are solvable. *P. E. Guenther* (Cleveland, Ohio)

SEQUENCES, SERIES, SUMMABILITY

See also 939, 1149.

1130:

Griesel, Heinz. Von den Folgen zu den Filtern. Math.-Phys. Semesterber. **6** (1958), 31–45.

An expository article.

1131:

Kuttner, B. On differences of fractional order. Proc. London Math. Soc. (3) **7** (1957), 453–466.

Given a sequence $\{a_n\}$ of complex numbers, the difference sequence of order s, s real, is defined by

$$\Delta^s a_n=\sum_{m=0}^{\infty}\binom{-s-1+m}{m}a_{n+m},$$

provided the series converges. A systematic study of these differences has been made by A. F. Andersen in his Copenhagen dissertation of 1921 and in later papers. The author proves a number of extensions of Andersen's results. Suppose $s\geqq\sigma>-1$, σ not an integer. Then if $\Delta^\sigma a_n$ is defined, $\Delta^s a_n=o(n^{\sigma+1})$, which is a best possible result. Under the same assumptions, $\Delta^r(\Delta^s a_n)=\Delta^{r+s}a_n$ holds for $s\geqq\sigma$, $r\geqq\sigma$, $r+s\geqq\sigma$. Further, the series defining the left member of this equation is summable (C, k) to the right member, provided $s\geqq\sigma$, $r+s>\sigma$, $k\geqq\sigma-r$, $k\geqq0$. If $s>-1$, $r\geqq0$, equality holds whenever the expression on the right exists. The same conclusion holds when both sides exist if $s>-1$, $r+s>-1$, and $r+s$ is not an integer. *E. Hille* (New Haven, Conn.)

1132:

Zablockaya, N. I.; and Smygun, A. N. Almost geometric progressions. Grodnen. Gos. Ped. Inst. Uč. Zap. Ser. Mat. **2** (1957), 55–59. (Russian)

A sequence $\{u_n\}$ of positive numbers for which $\lim u_{n+1}/u_n=q$ is called an "almost geometric progression". Letting s_n denote partial sums, it is proved that if $1\leqq q\leqq\infty$, then $\lim u_n/s_n=(q-1)/q$ (=1 for $q=\infty$).

1133:

Kracovskaya, E. S. On a class of sequences. Grodnen. Gos. Ped. Inst. Uč. Zap. Ser. Mat. **2** (1957), 79–85. (Russian)

An increasing sequence $\{u_n\}$ of positive numbers for which $A\leqq nu_n/s_n\leqq B$, for positive A, B is called an "almost arithmetic progression". It is proved that if $\lim u_{2n}/u_n=c\geqq1$, then

$$c^{-2}\leqq\liminf N\leqq\limsup N\leqq c^2[1+2(c-c^{-1})],$$

where N denotes nu_n/s_n.

1134:

Rutishauser, Heinz. Beschleunigung der Konvergenz einer gewissen Klasse von Kettenbrüchen. Z. Angew. Math. Mech. **38** (1958), 187–190.

"Es wird gezeigt, wie eine gewisse Klasse von schlecht konvergenten Kettenbrüchen durch iterierte Anwendung der Transformation von Bauer und Muir so umgeformt werden können, daß sie asymptotisch wie eine geometrische Reihe mit dem Quotienten — 0,17157··· konvergieren." (Zusammenfassung des Autors)

A. Edrei (Syracuse, N.Y.)

1135:

Orlicz, W. On the continuity of linear operations in Saks spaces with an application to the theory of summability. Studia Math. **16** (1957), 69–73.

Let X_0 be the union of linear subspaces $X_1, X_2, \cdots$ of a space X. Conditions involving these spaces and a sequence of operations $U_1, U_2, \cdots$ are given which imply that $U_n(x_n)\to0$ whenever $x_1, x_2, \cdots$ is a sequence satisfying stated conditions. A similar theorem gives conditions which imply that an operation U is continuous, and a third involves separability of a set of sequences. These theorems, of which the statements are lengthy, have applications to the theory of matrix methods for evaluation of sequences. If $A^{(1)}, A^{(2)}, \cdots$ are regular for null sequences, and if the set E of bounded divergent sequences evaluable to 0 by each one of these methods $A^{(n)}$ is nonempty, then E is nonseparable in the space B of bounded sequences. *R. P. Agnew* (Ithaca, N.Y.)

1136:

Albrecht, Rudolf. Auswahlverfahren bei linearer Limitierung. Math. Z. **67** (1957), 320–331.

Let C be a nonempty countable collection of square matrices defining matrix methods for evaluation of sequences. In terms of a separation of the sequence 0, 1, 2, ··· into a countable collection of disjoint infinite subsequences, a rule is given for the construction of a matrix A of which each row consists of an interspersion of zeros and the elements of one of the rows of one of the matrices in the collection C. The elements of the A transform of a sequence $s_0, s_1, \cdots$ are then elements of transforms of subsequences by matrices in the collection C. Several problems, involving evaluability of given sequences and regularity of methods, are solved.

R. P. Agnew (Ithaca, N.Y.)

1137:

Volkov, I. I. Some questions of linear matrix transformations. Mat. Sb. N.S. **44(86)** (1958), 85–112. (Russian)

In the theory of matrix transformations, it is usually said that a given sequence $s_0, s_1, \cdots$ of complex numbers is evaluable to T by a matrix a_{nk} of complex numbers if the series in $T_n=\sum_{k=0}^{\infty}a_{nk}s_k$ converges for each n, and $T_n\to T$ as $n\to\infty$. Without requiring that these series be convergent, let the sequence s_n be said to be essentially evaluable to $T'+iT''$ if the real and imaginary parts of $\sum_{k=0}^{q}a_{nk}s_k$ have, as $q\to\infty$, inferior and superior limits which approach T' and T'' as $n\to\infty$. An example shows that a sequence may be essentially evaluable even when

all of the series $\sum a_{nk}s_k$ are divergent. A matrix a_{nk} is essentially regular if each convergent sequence is essentially evaluable to the value to which it converges. Necessary and sufficient conditions for essential regularity are almost the same as the Toeplitz conditions for regularity, the norm condition being $\sum_{k=0}^{\infty}|a_{nk}|<M$ for each sufficiently great n but not necessarily for each n. The remaining results involve real matrices and sequences. There is no regular matrix by which each sequence $s_0, s_1, \cdots$ for which $s_n\to\infty$ is essentially evaluable to $-\infty$. A matrix a_{nk} is said to be essentially totally equivalent to convergence if s_n is essentially evaluable to T, where T may be finite or $+\infty$, if and only if $a_n\to T$. It is shown that if a_{nk} is essentially totally equivalent to convergence, then there is an index K such that to each $k>K$ there corresponds an index $n(k)$ such that $a_{n(k),k}\neq 0$ and $a_{n(k),j}=0$ when $j\geq K$ and $j\neq k$.

R. P. Agnew (Ithaca, N.Y.)

1138:

Davydov, N. A. On (s)-points of a sequence summable by the Poisson-Abel method. Mat. Sb. N.S. **43(85)** (1957), 67–74. (Russian)

Let $a_0+a_1+\cdots$ be a series of complex numbers for which $\limsup|a_n|^{1/n}=1$ and let $S_n=a_0+a_1+\cdots+a_n$. A complex number A is called an (s)-point of the sequence S_n if to each $\varepsilon>0$ there correspond a number $\lambda(\varepsilon)>1$ and sequences m_k and n_k of integers such that $n_k\to\infty$, $m_k/n_k\geq\lambda(\varepsilon)$, and $|S_n-A|<\varepsilon$ when $n_k<n\leq m_k$. Supplementing known results involving Cesàro and Abel evaluability of sequences having (s)-points, the author proves the following theorem. To each closed set F in the complex plane there corresponds a sequence which is evaluable by the Abel method and for which F is the set of (s)-points. Other closely related results are given.

R. P. Agnew (Ithaca, N.Y.)

1139:

Rosenblum, Marvin. On the Hilbert matrix. I. Proc. Amer. Math. Soc. **9** (1958), 137–140.

T. Kato [same Proc. **8** (1957), 73–81; MR **18**, 786] showed first that π is a characteristic root of the Hilbert matrix $((n+m+1)^{-1})$. More generally he showed that the matrix $H_k=((n+m+1-k)^{-1})$ has the characteristic root π when $\frac{1}{2}\geq k$. Here it is shown that every complex number with positive real part is a characteristic root of H_k, $k<1$. The corresponding characteristic vector is given explicitly. It is $\{x_n\}$, where $x_n=\int_0^\infty f(t)\phi_n(t)dt$, with $\phi_n(x)$ the normalized nth Laguerre polynomial and $f(x)=W_{k,n}(x)x^{-1}$, where $W_{k,m}$ is the Whittaker function. Kato had shown that the eigenvector corresponding to π for H_0 can be chosen positive and enquired about generalizations of this fact. Here a more general result is given showing the exceptional role of $k=\frac{1}{2}$. Further, it was known that the characteristic vector corresponding to π in H_0 could not be taken in l^2. This fact, too, is generalized.

O. Taussky-Todd (Pasadena, Calif.)

1140:

Jakimovski, Amnon. An inclusion property for Hausdorff means. Riveon Lematematika **10** (1956), 37–40. (Hebrew. English summary)

In the terminology of G. H. Hardy [Divergent series, Oxford, 1949; MR **11**, 25], the author proves that a regular Hausdorf mean $[H,\varphi]$ includes the Euler mean (E,q) of order $q\geq 0$ if and only if $\varphi(t)$ is constant in the interval $(1+q)^{-1}\leq t\leq 1$. *A. Dvoretzky* (Princeton, N.J.)

1141:

Chen, Kien-Kwong. Convergence of absolutely summable series. Sci. Sinica **4** (1955), 211–228.

A series $u_0+u_1+\cdots$ with partial sums $s_0, s_1, \cdots$ is said to be absolutely evaluable A, or evaluable $|A|$, if the series in $f(x)=\sum u_kx^k$ converges when $0\leq x<1$ and f has bounded variation over this interval. The series is evaluable $|C_1|$ if $\sum|\sigma_{n+1}-\sigma_n|<\infty$ where

$$\sigma_n=(n+1)^{-1}\sum_0^n s_k.$$

It is well known that each series evaluable $|C_1|$ is also evaluable $|A|$. The purpose of this paper is to prove the following Tauberian proposition. If $\sum u_n$ is evaluable $|A|$, if the sequence s_n is bounded, and if $u_n\to 0$ as $n\to\infty$, then $\sum u_n$ must be convergent. {The proposition is false. The following is an example of a divergent series $\sum u_n$, such that s_n is bounded and $u_n\to 0$, which is evaluable $|C_1|$. Let $s_n=0$ when $n<4!$. For each $q=4, 5, 6, \cdots$, let $s_{q!+k}=\sin(\pi k/2q)$ when $0\leq k\leq 4q$ and let $s_n=0$ when $q!+4q<n<(q+1)!$. The series $\sum u_n$ having partial sums s_n is divergent, $|s_n|\leq 1$, and $u_n=s_n-s_{n-1}\to 0$ as $n\to\infty$. The estimate

$$\sum_{n=q!+1}^{(q+1)!}|\sigma_n|=\sum_{n=q!+1}^{q!+4q}\left|\frac{s_{q!+1}+s_{q!+2}+\cdots+s_n}{n+1}\right|<\sum_{n=q!+1}^{q!+4q}\frac{4q}{q!}=\frac{(4q)^2}{q!}$$

shows that $\sum|\sigma_n|<\infty$, and hence that $\sum|\sigma_{n+1}-\sigma_n|<\infty$, so the series is evaluable $|C_1|$.}

R. P. Agnew (Ithaca, N.Y.)

1142:

Yano, Kenji. A Tauberian theorem on the Cesàro summation method. Sûgaku **9** (1957/58), 151–153. (Japanese)

Let σ_n^α denote the Cesàro means of $\sum a_n$ of order $\alpha>-1$. The author shows by use of a result of O. Szász [Proc. Amer. Math. Soc. **3** (1952), 257–263; MR **13**, 835] that if $\sum a_n$ is Abel summable to s and $\alpha>-1$, then $\sigma_n^\alpha\to s$ under the following conditions:

$$(1)\qquad \frac{\sigma_{n+m}^{\alpha+1}-\sigma_n^{\alpha+1}}{\log(n+m)-\log n}=o(1)\ (n\to\infty),$$

$$(2)\qquad \sigma_{n+\nu}^\alpha-\sigma_n^\alpha>-\varepsilon_n\ (\nu=1, 2, \cdots, m),$$

where $m=m(n)$ is of lower order of infinity than n, and ε_n is a sequence of positive terms tending to zero as $1/n$.

S. Ikehara (Tokyo)

APPROXIMATIONS AND EXPANSIONS

See also 937, 1173, 1402.

1143:

Eremine, S. A. Les évaluations asymptotiques des meilleures approximations de certaines fonctions de deux variables. Ukrain. Mat. Ž. **9** (1957), 413–418. (Russian. French summary)

For the product $f_1(z)f_2(w)$, $-1\leq z,w\leq 1$, the author writes a Newton interpolation formula whose main term is a polynomial of degree n_1 in z and of degree n_2 in w. The remainder contains certain line integrals which appear also for a function of one variable. Estimates for these obtained by S. Bernstein [Leçons sur les propriétés extrémales, Gauthier-Villars, Paris, 1926] and Ibragimov

[Izv. Akad. Nauk SSSR. Ser. Mat. **10** (1946), 429–460; MR **8**, 459] allow one to give upper estimates for the degree of approximation when each of the functions $f_1(z)$, $f_2(z)$ is of the form $(a-z)^l \log^k (a-z)$.

G. G. Lorentz (Syracuse, N.Y.)

1144:

Radu, N. C. Sur une formule d'interpolation. Com. Acad. R. P. Romîne **7** (1957), 693–696. (Romanian. Russian and French summaries)

The author starts from an interpolation formula attributed to O. Onicescu: given $n+1$ interpolation points, form the Lagrange polynomials for f with all selections of $p+1$ of these points, and form a linear combination of these with coefficients of sum 1, thus obtaining a new approximating polynomial for f. The author extends this idea by taking into account not only f but also a prescribed number of its derivatives.

R. P. Boas, Jr. (Evanston, Ill.)

1145:

Tchakaloff, Vladimir. Formules de cubatures mécaniques à coefficients non négatifs. Bull. Sci. Math. (2) **81** (1957), 123–134.

Given a two-dimensional domain F the author proves that one can always find a formula for integration over this domain which possesses positive coefficients and which is exact for polynomials up to a certain degree. Specifically, let F be a closed bounded set of the Euclidean u, v plane with positive two-dimensional Borel-Lebesgue measure. For each integer $n \geqq 0$, let $N=\frac{1}{2}(n+1)(n+2)$, and let E_N be the space of all polynomials $\sum_{p+q \leq n} a_{pq} u^p v^q$ of degree $\leqq n$. Then there exists N points $[u_k, v_k]$ $(k=1, \cdots, N)$ belonging to F, and N constants $a_k \geqq 0$, such that

$$\iint_F x(u, v)\, d\omega = \sum_{k=1}^{N} a_k x(u_k, v_k)$$

for all $x \in E_N$. A generalization [due fundamentally to F. Riesz, Ann. Sci. École Norm. Sup (3) **28** (1911), 33–62] giving a representation for positive functionals over a vector space E_N is first proved and the above result is deduced from it. The author remarks that a similar result for p-dimensional integrals with $N=\binom{n+p}{p}$ can be proved by the same technique.

P. J. Davis (Washington, D.C.)

1146:

***Mergelyan, S. N. Weighted approximations by polynomials.** American Mathematical Society Translations, Ser. 2, Vol. 10, pp. 59–106. American Mathematical Society, Providence, R. I., 1958. iv+409 pp. $6.60.

Translated from Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5 (71), 107–152; MR **18**, 734.

1147:

Freud, Géza. Über die Asymptotik orthogonaler Polynome. Acad. Serbe Sci. Publ. Inst. Math. **11** (1957), 19–32.

Let $f(\theta)$ be a weight function on the unit circle $-\pi < \theta \leqq \pi$, satisfying the following conditions: (a) $f(\theta)$ is measurable; (b) $0<m \leqq f(\theta) \leqq M$; (c) with a fixed γ we have $|f(\theta)-f(\gamma)|<A|\theta-\gamma|$. Denoting the orthonormal polynomials associated with $f(\theta)$ on the unit circle by $\{\varphi_n(z)\}$, $z=e^{i\theta}$, we have for $z=e^{i\gamma}$

$$\varphi_n(z)=z^n\{\overline{D(z)}\}^{-1}+o(1) \quad (n \to \infty),$$

where $D(z)$ denotes the (necessarily existing) boundary values of a certain analytic function associated with $f(\theta)$ in a well-known manner. This formula, known originally by the works of S. Bernstein and G. Szegö, was proved recently under more general conditions by Geronimus; the present paper uses the still more general conditions mentioned above, and represents further progress in the same direction.

G. Szegő (Stanford, Calif.)

1148:

***Качмаж. С.; и Штейнгауз, Г. Теория ортогональных рядов. [Kačmaž, S. (Kaczmarz, S.); and Steingauz, G. (Steinhaus, H.). Theory of orthogonal series.] Translation from the German with a summary of later results by R. S. Guter and P. L. Ul'yanov. Edited, with supplements, by N. Ya. Vilenkin.** Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 507 pp. (1 insert). 17.50 rubles.

The translators have added 50% new material to this excellent German text. Since it has not been reviewed in MR, we include a brief summary of the original chapters. The book deals with orthonormal systems $\{\varphi_n\}$ of functions on a finite interval; the trigonometric system is not discussed at all, other concrete systems only as examples. Chapter I gives preliminary results about integration, functionals and operators. Ch. II–IV are relatively elementary and deal with basic notions, the theory of orthogonal series in L^2 (with convergence in norm), and examples. Ch. V deals with convergence and summability almost everywhere of orthogonal series in L^2 and centers around the Menchoff-Rademacher theorem. Ch. VI, orthogonal series in other spaces (such as L^p, V, etc.), discusses completeness and closure, theorems of F. Riesz and Paley with counterexamples, multiplicators. Ch. VII deals with "lacunary" systems, for which $p>2$ exists such that $\sum a_n^2<+\infty$ implies the convergence in L^p norm of $\sum a_n \varphi_n$. Ch. VIII contains some results about biorthogonal systems and systems of orthogonal polynomials.

The very useful summary of new results by Guter and Ul'yanov contains many detailed proofs. Following topics are treated. Conditions for completeness of a system; divergence a.e. of $\sum \varphi_n^+(x)^2$ for complete systems. Marcinkiewicz's results about convergence of a subsequence $S_{m_i}(x)$ of partial sums of an orthogonal series, with m_i depending only on the φ_n; properties of a system which can be strengthened by rearrangement. "Universal" orthogonal series [Talalyan, an unpublished dissertation] generalize results obtained by Menchoff for the trigonometric system. Unconditional convergence and absolute convergence (results by Stečkin), summability of orthogonal series by Cesàro means, C_1-summability of lacunary series. Results about series for which the sum of $\sum \varepsilon_n a_n \varphi_n(x)$ belongs to a class A for all possible $\varepsilon_n=\pm 1$, or all possible $|\varepsilon_n| \leqq 1$. Stable properties of a system $\{\varphi_n\}$, which hold also for each system $\{\psi_n\}$ (after a change on a set of measure zero) close to $\{\varphi_n\}$ in the sense $\rho_n^2=\int_a^b (\varphi_n-\psi_n)^2 dx \to 0$, or in the sense $\sum \rho_n<+\infty$. Bases in the space L^2, in particular "Bessel" bases and "Hilbert" bases, for which one or the other part of the Riesz-Fischer theorem holds, with their characterization; almost orthogonal systems with the property

$$\sum_{n \neq m} \left(\int_a^b \varphi_n \varphi_m dx\right)^2 < +\infty.$$

The supplement by Vilenkin deals with multiplicative (complex-valued) systems $\{\varphi_n\}$, which have the property that $\varphi_n \varphi_m$ and $1/\varphi_n$ $(n, m=1, 2, \cdots)$ belong to the system. Haar's theory of such systems can be made more natural by the use of a representation of the φ_n as characters on a compact abelian group. Vilenkin's results about periodic

multiplicative systems are given, for which integers m_n exist with $\varphi_n(x)^{m_n}\equiv 1$. These include the system of Walsh and other related systems.

G. G. Lorentz (Syracuse, N.Y.)

1149:

Vermes, Robert. Abel's limit theorem and Tauber's theorem in the series of orthogonal polynomials. Riveon Lematematika **10** (1956), 72–80. (Hebrew. English summary)

The author proves analogues of the Abel continuity theorem and of the original Tauber theorem for series $\sum a_n p_n(z)$, where $p_n(z)$ are the Szegö orthogonal polynomials associated with a simple closed analytical curve in the complex plane [Math. Z. **9** (1921), 218–270]. The proof depends on the asymptotic properties of the zeros of $p_n(z)$. *A. Dvoretzky* (Princeton, N.J.)

1150:

Standish, Charles. The L_p convergence of Fourier-Bessel series for $0<p<1$. Proc. Amer. Math. Soc. **8** (1957), 484–486.

The author proves that if $f\in L_1(0, 1)$ and $\nu\geqq -\frac{1}{2}$ then the J_ν Fourier-Bessel series for f converges to f in L_p for $0\leqq p\leqq 1$. *L. H. Loomis* (Cambridge, Mass.)

1151:

Voronovskaya, E. V. On the closest uniform approximation of polynomials. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 927–929. (Russian)

A polynomial $Q_n(x)$ with $\max_{0,1}|Q_n(x)|=1$ is called of class II and of passport $\pi=[n, s, p]$ if n is its degree, s the number of points $\sigma_1<\sigma_2<\cdots<\sigma_s$ of $[0, 1]$, where $\max|Q_n(x)|$ is attained, p the number of $i=1, \cdots, s-1$ for which $Q_n(\sigma_i)$ and $Q_n(\sigma_{i-1})$ are of the same sign, if in addition $s>\frac{1}{2}n+1$. The author describes properties of polynomials Q_n of a given passport similar to those developed for the special case $\pi=[n, n-1, 0]$ previously [same Dokl. (N.S.) **110** (1956), 727–730; MR **18**, 730], and gives means for their determination.

G. G. Lorentz (Syracuse, N.Y.)

1152:

Bram, Joseph. Chebychev approximation in locally compact spaces. Proc. Amer. Math. Soc. **9** (1958), 133–136.

A characterization of "polynomials" of best approximation to a continuous function in terms of determinants of a certain matrix. This result appears also in a 1935 book in Ukrainian by E. Remez, and in Žuhovickiĭ, Uspehi Mat. Nauk (N.S.) **11** (1956) no. 2(68), 125–159 [MR **19**, 30]. *G. G. Lorentz* (Syracuse, N.Y.)

1153:

Baskakov, V. A. An instance of a sequence of linear positive operators in the space of continuous functions. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 249–251. (Russian)

The author considers a sequence of operators $L_n(f; x)=\sum_{k=0}^{\infty}(-1)^k\phi_n^{(k)}(x)x^k f(k/n)/k!$, where the functions ϕ_n are analytic and satisfy conditions which guarantee that for each continuous f, $L_n(f; x)$ shall converge uniformly to $f(x)$. For example, the conditions are satisfied when $\phi_n(x)=(1-x)^n$, in which case $L_n(f; x)$ is the Bernstein polynomial of order n. The theorems stated in the paper, without proof, provide estimates for $L_n(f; x)-f(x)$.

M. Jerison (Princeton, N.J.)

1154:

Mishoe, Luna I. On the Gibb's phenomenon in a certain eigenfunction series. Proc. Amer. Math. Soc. **9** (1958), 1–5.

Let $u_n(x)$ be the eigenfunctions of the equation

$$u''+q(x)u+\lambda[p(x)u-u']=0$$

under the boundary conditions $u(0)=u(1)=0$ where $q(x)$ and $p''(x)$ are continuous on $(0, 1)$; and let $v_n(x)$ be the eigenfunctions of the adjoint system. The eigenfunction expansion of a function $f(x)$ of bounded variation is $\sum_{-\infty}^{\infty}a_n u_n(x)$ where, with B^* the operator $d/d\xi+p(\xi)$, $a_n=(\int_0^1 u_n(\xi)B^*v_n(\xi)d\xi)^{-1}(\int_0^1 f(\xi)B^*v_n(\xi)d\xi)$. It was shown that this expansion converges to $\frac{1}{2}f(x+)+\frac{1}{2}f(x-)+c\exp\int_0^x p(t)\,dt$, $c=-\frac{1}{2}f(0+)-\frac{1}{2}f(1-)\exp\int_0^1 -p(t)dt$. [B. Friedman and L. I. Mishoe, Div. Elec. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **BR-4** (1954); Pacific J. Math. **6** (1956), 249–270; MR **15**, 796; **18**, 129].

It is now proved that the partial sum

$$\sigma_n(x)=\sum_{k=-n}^{n}a_k u_k(x)$$

exhibits Gibbs' phenomenon for the function $g(x)=f(x)+c\exp\int_0^x p(t)dt$ whenever the Fourier partial sum

$$S_n(x)=\sum_{k=-n}^{n}e^{2k\pi ix}\int_0^1 f(\xi)e^{-2k\pi i\xi}d\xi$$

exhibits the Gibbs' phenomenon for $f(x)$.

E. T. Copson (St. Andrews)

FOURIER ANALYSIS

See also 1182, 1213, 1349.

1155:

Ogiewetzki, I. I. Generalization of the inequality of P. Civin for the fractional derivative of a trigonometrical polynomial to L_p space. Acta Math. Acad. Sci. Hungar. **9** (1958), 133–135. (Russian summary)

The inequality $\|T_n^\alpha\|_p\leqq C(\alpha)n^\alpha\|T_n\|_p$, $p\geqq 1$ for the fractional derivatives of order α of a trigonometric polynomial T_n is proved. *G. G. Lorentz* (Syracuse, N.Y.)

1156:

Menchoff, D. E. On the limit functions of a trigonometric series. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 476–478. (Russian)

The function $\varphi(x)=\varphi(x, E)$, defined almost everywhere on the set $E\subset[-\pi, \pi]$, meas $E>0$, is said to be a limit function of the series $\sum u_n(x)$ if there exists a sequence of integers $\rho_1<\rho_2<\cdots$ such that $\varphi(x)=\lim_{k\to\infty}\sum_{\rho_k}^{\rho_{k+1}}u_n(x)$, for almost all $x\in E$. If $E=\liminf E_n$, meas $E>0$, meas $E'>0$, meas $(E'-E)=0$, and if $\varphi(x, E')=\lim\varphi_n(x, E_n)$ for almost all $x\in E'$, then $\varphi(x, E')$ is said to be a limit function in the wide sense of the sequence $\varphi_n(x, E_n)$. A set $M=\{\varphi_\alpha(x, E_\alpha)\}$ is said to be closed in the restricted sense if it contains all the functions $f(x, E')$ which are limit functions of sequences $\{\varphi_n(x, E_n)\}\subset M$.

Theorem 1. The set $M=\{\varphi_\alpha(x, E_\alpha)\}$ is closed in the restricted sense if and only if there exists a trigonometrical series $a_0/2+\sum(a_n\cos nx+b_n\sin nx)$ such that M is the set of all its limit functions. The theorem is derived from two general results concerning limit functions of arbitrary series $\sum u_n(x)$. No proofs are given.

M. Cotlar (Buenos Aires)

1157:

Uljanov, P. L. On divergence of Fourier series. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **8** (1958), 259–324. (Hungarian)

A translation of the article in Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 75–132 [MR **19**, 854], with minor corrections by D. Králik.

1158:

Misra, M. L. On the summability of a Fourier series and its conjugate series. Bull. Allahabad Univ. Math. Assoc. **17** (1956/57), 4–17 (1958).

Summary of a thesis presented to the Allahabad University in 1947. Eleven theorems are proved, 1 on convergence of conjugate series, 4 on Cesàro summability of various related series, 3 on Abel summability, and 3 on Riesz summability.

1159:

Singh, U. N. Strong summability of trigonometrical series. Bull. Allahabad Univ. Math. Assoc. **17** (1956/57), 31–39 (1958).

Summary of a thesis presented to the Allahabad University in 1949.

1160:

Siddiqi, J. A. The Norlund summability of trigonometrical series. Bull. Allahabad Univ. Math. Assoc. **17** (1956/57), 40–50 (1958).

Summary of a thesis presented to the Allahabad University in 1949.

1161:

Pati, Tribikram. Contributions to the study of absolute summability of series. Bull. Allahabad Univ. Math. Assoc. **17** (1956/57), 18–30 (1958).

Summary of a thesis presented to the Allahabad University in 1956. The series in question are Fourier or closely related series.

1162:

Kiyohara, Mineo. On the local property of the absolute summability $|C, \alpha|$ for Fourier series. J. Math. Soc. Japan **10** (1958), 55–63.

In 1932 N. Wiener proved that the Fourier series of a function $f(x)$ periodic and Lebesgue integrable in $(-\pi, \pi)$ converges absolutely, if to every x_0 corresponds a function $g(x, x_0)$ with an absolutely convergent Fourier series such that $g(x, x_0)$ coincides with $f(x)$ in the neighborhood of x_0. In 1956, V. Magarik [Moskov. Gos. Univ. Uč. Zap. **181** Mat. **8**, 183–196; MR **19**, 31] extended this result to the absolute summability $|C, \alpha|$, $\alpha \geqq 0$, but his generalization remained incomplete: to the assumption of absolute summability $|C, \alpha|$ of the Fourier series of $g(x, x_0)$ was added that of its conjugate series.

The author, using general properties of $|C, \alpha|$, succeeds in eliminating this supplementary assumption, thus obtaining a complete generalization of Wiener's theorem to $|C, \alpha|$. *E. Kogbetliantz* (New York, N.Y.)

1163:

Rudin, Walter. Remarks on a theorem of Paley. J. London Math. Soc. **32** (1957), 307–311.

The author is concerned with the following interesting question: if $\phi \in L(-\pi, \pi)$ has Fourier coefficients $\{c_n\}$ $(-\infty < n < \infty)$, how arbitrary can be the behaviour of a preassigned subsequence $\{c_{n_k}\}$, where $n_k \uparrow \infty$? He obtains several loosely connected results. The starting point is a theorem of Paley [Ann. of Math. (2) **34** (1933), 615–616]. Let $\phi \in H_1$, the subspace of functions for which $c_n = 0$ for $n < 0$; one may set $f(z) = \sum_0^\infty c_n z^n$, obtaining a function analytic in $|z| < 1$ for which $\int_{-\pi}^{\pi} |f(re^{i\theta})| d\theta$ is uniformly bounded for all $r < 1$. It is not in general true that $\sum |c_n|^2$ converges, but for any choice of $\phi \in H_1$, $\sum |c_{2^n}|^2 < \infty$; more generally, Paley showed that $\sum |c_{n_k}|^2$ converges for any sequence $\{n_k\}$ having Hadamard gaps, $n_{k+1}/n_k \geqq q > 1$. The author first observes that this is essentially best possible; a sequence $\{n_k\}$ is a Paley sequence only if it is the union of a finite number of Hadamard type sequences. Now consider a general sequence $\{a_k\}$ with $\sum |a_k|^2 < \infty$, and ask if there is a function ϕ whose Fourier coefficients obey $c_n = a_k$ when $n = n_k$, and $c_n = 0$ for $n \geqq 0$ with $n \neq n_k$. Reducing this to the previous condition, the author shows that if this holds with ϕ bounded, $\{n_k\}$ must have been a Paley sequence, and that in this case, ϕ can in fact always be chosen to be continuous, or to have constant absolute value. Finally, the author employs an approach used by Helson [Proc. Amer. Math. Soc. **6** (1955), 235–242; MR **17**, 597] to show that the analogue of the Paley theorem fails for sums $\sum |c_{n_k}|$ even when ϕ is restricted to the subspace A of functions which arise as boundary values of functions continuous on $|z| \leqq 1$ and analytic in $|z| < 1$. *R. C. Buck* (Madison, Wis.)

1164:

***Ivaševˇ-Musatov, O. S. On Fourier-Stieltjes coefficients of singular functions.** American Mathematical Society Translations, Ser. 2, Vol. 10, pp. 107–124. American Mathematical Society, Providence, R. I., 1958. iv+409 pp. $6.60.

Translated from Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 179–196; MR **17**, 1081.

1165:

Lebed', G. K. Inequalities for polynomials and their derivatives. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 570–572. (Russian)

In this paper the author gives estimates of the derivatives of the trigonometric polynomial $T_n(\theta) = \sum_{k=0}^{n} (a_k \cos k\theta + b_k \sin k\theta)$ and the algebraic polynomial $P_n(x) = \sum_{k=0}^{n} a_k x^k$. The results are presented in a summarized form by a few theorems.

S. Kulik (Logan, Utah)

1166:

Maak, Wilhelm. Über eine Verallgemeinerung der doppeltperiodischen Funktionen. Abh. Math. Sem. Univ. Hamburg **21** (1957), 104–108.

Let $\mathfrak{G}$ be the group of operators $\mathfrak{t}_n$, n an integer, defined by $\mathfrak{t}_n f(s) = f(s + in)$ where $s = \sigma + it$ is a complex number in a strip $\alpha < \sigma < \beta$. Define the norm of $f(s)$ by $\|f\| = \sup_{\alpha < \sigma < \beta} |f(s)|$. Then the set of bounded regular functions $f(s)$ having a bounded norm form a function space. The functions $f(s)$ which are "almost periodic points" of this space with respect to the group $\mathfrak{G}$ are the almost periodic functions in the strip (α, β) according to the definition of H. Bohr. In the present paper the above definition is extended in the obvious way to define almost doubly periodic functions in a parallelogram, and properties of these functions are obtained. Typical of the results obtained is the following: the module $\mathfrak{A}$ of all doubly almost periodic functions is the smallest closed module which contains all irreducible submodules of $\mathfrak{A}$.

W. H. Simons (Vancouver, B.C.)

1167:

Burkill, H. The Cesàro-Perron scale of almost periodicity. Proc. London Math. Soc. (3) **7** (1957), 481–497.

Soit $f(x)$ une fonction à valeurs complexes, défini pour $-\infty<x<+\infty$; on suppose que l'intégrale répétée

$$f_{r+1}(x)=\int_0^x dx_1\int_0^{x_1}dx_2\cdots\int_0^{x_r}f(t)dt$$

existe au sens Cesàro-Perron [voir H. Burkill, Proc. London Math. Soc. (3) **2** (1952), 150–174; MR **14**, 162], $f_{r+1}(x)$ est alors continue.

On considère ensuite l'espace vectoriel normé des fonctions telles que

$$\|f\|=\sup_{0\le h\le 1}\left|\int_x^{x+h}dx_1\int_{x_1}^{x_1+h}dx_2\cdots\int_{x_r}^{x_r+h}f(t)dt\right|=$$
$$\sup_{0\le h\le 1}\left|f_{r+1}[x+(r+1)h]-\binom{r}{1}f_{r+1}(x+rh)+\cdots\right|;$$

une fonction de cet espace est dite C_rP presque-périodique si, pour $\varepsilon>0$ donné quelconque, il y a un ensemble relativement dense de presque-périodes τ, telles que $\|f(x+\tau)-f(x)\|<\varepsilon$. On démontre alors que l'ensemble de ces fonctions est identique à la complétion, au sens C_rP, de l'ensemble des polynomes trigonométriques. Toute fonction C_rP p.p. a une moyenne de Cesàro d'ordre $(r+1)$, donnée par

$$\lim_{X\to\infty}\frac{r+1}{X^{r+1}}\int_0^X(X-x)^rf(x)dx=\lim_{X\to\infty}\frac{(r+1)!}{X^{r+1}}f_{r+1}(X);$$

de la résulte l'existence de la série de Fourier et la théorème d'unicité. On a ensuite des résultats du genre des suivants. (1) Pour que la fonction f, supposée C_rP intégrable, soit C_rP p.p., il faut et il suffit que

$$\Delta^{r+1}[f_{r+1}(x),h]=f_{r+1}[x+(r+1)h]-\binom{r}{1}f_{r+1}(x+rh)+\cdots$$

soit uniformément continue en h, pour $0\le h\le s$ et $-\infty<x<+\infty$, et que $\Delta^{r+1}[f_{r+1}(x),h]$ soit, pour tout h, une fonction continue p.p. (2) Si f est C_rP p.p., et si $f_{r+1}(x)$ est bornée, alors $f_1(x)$ est $C_{r-1}P$ p.p.

J. Favard (Paris)

1168:

Matveev, I. V. On summation of double Fourier series of functions of two variables. Uspehi Mat. Nauk (N.S.) 12 (1957), no. 5(77), 221–229. (Russian)

Let $f(x,y)$ be an integrable function of period 2π, a_{kl}, b_{kl}, c_{kl}, d_{kl} the Fourier coefficients of $f(x,y)$ (with usual modifications for $k=0$, $l=0$), and

$$U_{mn}(f;x,y)=\sum_{k=1}^{m}\sum_{l=1}^{n}\lambda_{kl}(a_{kl}\cos kx\cos ly+b_{kl}\sin kx\cos ly$$
$$+c_{kl}\cos kx\sin ly+d_{kl}\sin kx\sin ly),$$

where $\{\lambda_{kl}=\lambda_{kl}^{(m,n)}\}$ is a system of numbers. Let E be the set of the points (x,y) satisfying the conditions:

a) $$\lim_{\alpha,\beta\to 0}(\alpha\beta)^{-1}\int_0^\alpha\int_0^\beta|\varphi_{xy}(t,\tau)|dtd\tau=0,$$

b) $$\sup_\alpha\alpha^{-1}\int_0^\alpha dt\int_0^\pi|\varphi_{xy}|d\tau<M,$$

c) $$\sup_\beta\beta^{-1}\int_0^\beta d\tau\int_0^\pi|\varphi_{xy}|dt<M,$$

where

$$\varphi_{xy}(t,\tau)=f(x+t,y+\tau)+f(x+t,y-\tau)$$
$$+f(x-t,y+\tau)+f(x-t,y-\tau)-4f(x,y).$$

The author proves the following result. Theorem: If the coefficients λ_{kl} ($\lambda_{k,n+1}=\lambda_{m+1,l}=0$) satisfy the conditions:

1) $$\lim\lambda_{kl}=1\ (m,n\to\infty),\ |\lambda_{kl}|<L;$$

2) $$\sum_{k=0}^{m-1}\Big(\sum_{i=m-k}^{m}(m-k)/i\Big)|\Delta^2\lambda_{kk}|<L;$$

3) $$\sum_{l=0}^{n-1}\Big(\sum_{j=n-l}^{n}(n-l)/j\Big)\ |\Delta^2\lambda_{ll}|<L;$$

4) $$\sum_{k=0}^{m-1}\sum_{l=0}^{n-1}\Big(\sum_{i=m-k}^{m}(m-k)/i\Big)\Big(\sum_{j=n-l}^{n}(n-l)/j\Big)|\Delta^4\lambda_{kkll}|<L;$$

then $\lim U_{mn}(f;x,y)=f(x,y)$ $(m,n\to\infty)$ holds for every point (x,y) of E. The conditions of the theorem are satisfied for instance if $\lambda_{kl}=(1-k/(m+1))\ (1-l/(n+1))$ (Fejér), or $\lambda_{kl}=\cos(k\pi/(2m+1))\ \cos(l\pi/(2n+1))$ (Rogosinski-Bernstein), or $\lambda_{kl}=A_{m-k}^{(r)}A_{n-l}^{(r)}/A_n^{(r)}A_m^{(r)}$ (Cesàro).

M. Collar (St. Louis, Mo.)

1169:

Arens, Richard. The boundary integral of $\log|\varphi|$ for generalized analytic functions. Trans. Amer. Math. Soc. **86** (1957), 57–69.

Suppose G is a discrete abelian group, G_+ is a subsemigroup of G which generates G and which contains the identity element, A_0 is the algebra of all continuous functions ϕ on Γ (the character group of G) whose Fourier transforms have their support in G_+, and Δ is the maximal ideal space of A_0. Then Δ is the set of all (bounded?) homomorphisms of G_+ into the multiplicative semigroup of the complex numbers, and hence is a multiplicative semigroup; Γ is the Šilov boundary of Δ (relative to A_0). The functions $\phi\in A_0$, extended to Δ in the natural way, are the generalized analytic functions under consideration.

With each $\eta\in\Delta$ there is associated a measure m_η on Γ which "represents" η in a manner which reduces to the Poisson integral representation in the classical case (where Δ is the closed unit disc). The author defines, for $\phi\in A_0$, $\theta\in\Delta$ and $\eta\in\Delta$,

$$\phi(\theta;\eta)=-\int_\Gamma\log|\phi(\theta\alpha)|m_\eta(d\alpha),$$

and obtains results concerning the finiteness of $\phi(\theta;\eta)$ as well as inequalities involving $\phi(\theta;\eta)$. If G is archimedian ordered, he proves that $\phi(\theta;\eta)$ depends continuously on ϕ, for each fixed pair (θ,η).

W. Rudin (New Haven, Conn.)

1170a:

Reiter, H. Beiträge zur harmonischen Analyse. II. Math. Ann. **133** (1957), 298–302.

1170b:

Reiter, H. Contributions to harmonic analysis. III. J. London Math. Soc. **32** (1957), 477–483.

[Part I: Acta Math. **96** (1956), 253–263]; MR **19**, 290.] Let G be a locally compact commutative group and let $\hat{G}$ be its character group. For any subset $A\subset L^1(G)$, the annihilator of A in $\hat{G}$ (considered as a subset of $L^\infty(G)$) is called the hull of A, or, in the language of the author, the cospectrum of A. The cospectrum of A is thus the intersection of the null sets of the Fourier transforms of the functions in A. Any closed subset $B\subset\hat{G}$ is the cospectrum of the closed ideal $I\subset L^1(G)$ obtained as the annihilator of B in $L^1(G)$. If there is no other closed ideal in $L^1(G)$ of which B is the cospectrum, then B is said to be a set of uniqueness; otherwise it is a set of multiplicity. In these two notes the author proves that if $\hat{G}$ is a homomorphic image of a group $\hat{H}$ and if C is the inverse image in $\hat{H}$ of

the closed subset B of $\hat{G}$, then B and C have the same type. The first note deals with uniqueness and the second with multiplicity. *L. H. Loomis* (Cambridge, Mass.)

INTEGRAL TRANSFORMS

See also 1155, 1363.

1171:

Natanson, I. P. On a two-dimensional problem of moments. Vestnik Leningrad. Univ. Ser. Mat. Fiz. Him. **9** (1954), no. 5, 35–40. (Russian)

1172:

Kilpi, Yrjö. Über das komplexe Momentenproblem. Ann. Acad. Sci. Fenn. Ser. A. I. no. **236** (1957), 32 pp.

Let N be a linear operator with domain $D(N)$ dense in a Hilbert space and satisfying $D(N)\subseteq D(N^*)$. Let $\overline{N}$ be the restriction of the operator N^* to $D(N)$. The operator N is called normal if $\|Nf\|=\|\overline{N}f\|$ on $D(N)$. N is called hypermaximal normal if $\overline{N}=N^*$.

The first part of this paper is devoted to the problem of the extension of normal operators to hypermaximal normal operators. The first theorem notes the well-known fact that if N has a hypermaximal normal extension, then $H_1=(\overline{N}+N)/2$ and $H_2=(N-\overline{N})/2i$ have self-adjoint extensions. In the second theorem it is assumed that H_1 and H_2 have self-adjoint extensions. Under these conditions lengthy necessary and sufficient conditions are given that N has a hypermaximal normal extension. {The statement of this theorem is not completely clear, the proof not entirely convincing to this reviewer.} Theorem 3 states that if each of the operators H_k $(k=1, 2)$ has deficiency $(0, 0)$ or $(1, 1)$, then N has either one hypermaximal normal extension or an infinity of such extensions. The proof of theorem 3 depends on theorem 2.

The second part of the paper is devoted to a generalized problem of moments. Let $\{C_{m,n}; m, n=0, 1, \cdots\}$ be a sequence of complex constants. In order that

$$(1)\qquad C_{m,n}=\int Z^m\bar{Z}^n d\rho(Z),$$

where $d\rho(Z)$ is a non-negative bounded measure with spectrum in a closed set T in the complex plane, it is necessary and sufficient that the functional $\mu(Q)=\sum_{m,n} a_{m,n}C_{m,n}$ be non-negative whenever the polynomial $Q=\sum_{m,n} a_{m,n}Z^m\bar{Z}^n$ is non-negative on T. A second theorem states that a sequence $\{C_{m,n}\}$ has the form (1) if and only if there exists an element f in a Hilbert space and a normal operator N on this space so that

$$C_{m,n}=(N^m f|N^n f).$$

A somewhat deeper fact is that the representation (1) has either one solution $d\rho(Z)$ or else an infinity of solutions. {If the closed set T which contains the spectrum of $d\rho(Z)$ is not specified, then sharper and more general results have already been obtained by the reviewer [Trans. Amer. Math. Soc. **77** (1954), 455–480; MR **16**, 584].}

A. Devinatz (St. Louis, Mo.)

1173:

Geronimus, Ya. L. On certain estimates in the theory of Toeplitz forms and orthogonal polynomials. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 25–27. (Russian)

Consider a positive definite Toeplitz form $T_n=\sum_{i,k=0}^{n} c_{i-k}x_i\bar{x}_k$, $\Delta_n=|c_{i-k}|_0^n$. Put $h_n=\Delta_{n+1}/\Delta_n$, $\mu_n=h_n-\lim h_n$. The c_k have the form $c_k=(2\pi)^{-1}\int_0^{2\pi} e^{-ik\theta}d\sigma(\theta)$; if σ is absolutely continuous let $p=\sigma'$. The author summarizes the role of the quantities μ_n in the theory of the form T_n and of the associated orthogonal polynomials. He tabulates a variety of conditions on $p(\theta)$ and the estimates for μ_n which follow from them. Let $\omega_k(\delta; f)$ be the L^k integral modulus of continuity of f, and let

$$I(\delta)=\sup_{|h|\le\delta}\{(2\pi)^{-1}\int_0^{2\pi}|p(\theta+h)-p(\theta)|\,[p(\theta)]^{-1}d\theta\}.$$

For example, if $p(\theta)\ge m>0$, $\mu_n^{\frac{1}{2}}$ has the upper bound $C_1\omega_4(n^{-1}; \log p)+C_2\omega_2(n^{-1}; p)$ or $C_3\omega_1(n^{-1}; p)^{\frac{1}{2}}$. If $I(\delta)\to 0$ as $\delta\to 0$, then $\mu_n\le CI(n^{-1})$.

R. P. Boas, Jr. (Evanston, Ill.)

1174:

Narain, Roop. On chain of Laplace transforms. Univ. e Politec. Torino. Rend. Sem. Mat. **16** (1956–57), 429–432.

Writing $f(t)\doteqdot\Phi(s)=s\int_0^\infty e^{-st}f(t)dt$, and then considering the chain of transforms

$$f(t)\doteqdot\Phi_1(s),\quad t^{-\frac{1}{2}}\Phi_1(1/t)\doteqdot\Phi_2(s),$$
$$2(t/\pi)^{-\frac{1}{2}}\Phi_{n-1}(1/4t^2)\doteqdot\Phi_n(s)\text{ for }n>2$$

the author obtains $f(t^{2^{n-1}})\doteqdot(2/\pi^{\frac{1}{2}}s)\Phi_n(s^2/4)$.

The restriction $R(s)>s_0>0$, placed on s by the author when defining Φ, makes the statement of the theorem and its proof meaningless. *J. L. Griffith* (Kensington, Ont.)

1175:

Narain, Roop. On a generalization of Hankel transform and self-reciprocal functions. Univ. e Politec. Torino. Rend. Sem. Mat. **16** (1956–57), 269–300.

This paper discusses the transform

$$g(x)=2^{-\nu}\int_0^\infty (xy)^{\nu+\frac{1}{2}}\chi_{\nu,k,m}(x^2y^2/4)f(y)dy,$$

where the most concise expression for $\chi_{\nu,k,m}(x)$ appears to be

$$x^{-\nu}G^{2,1}_{2,4}\left(x\,\middle|\,\begin{matrix}k-m-\frac{1}{2}, \nu-k+m+\frac{1}{2}\\ \nu, \nu+2m, -2m, 0\end{matrix}\right).$$

When $k+m=\frac{1}{4}$, the transform reduces to

$$g(x)=\int_0^\infty (xy)^{\frac{1}{2}}J_\nu(xy)f(y)dy.$$

An inversion theorem is obtained and two separate discussions of self-reciprocal functions are given. The first of these follows closely the relevant parts of Chapter IX of Titchmarsh, "fourier integrals", Clarendon Press, Oxford, 1948, and the second expresses results in terms of Whittaker transforms [Varma, Proc. Nat. Acad. Sci. India Sect. A **20** (1951), 209–216; MR **15**, 868].

{In the proof of Lemma 1, the author assumes that if $\int_0^\infty e^{-ky}F(y)dy=0$, then $F(y)=0$.

On p. 277, we find $R(\nu+\frac{1}{2})>R(\mu+s)>0$, and in the following paragraph $\mu=\nu+\frac{1}{2}$. This shows that $R(s)<0$. However it is clear from p. 275 that $R(s)>0$.

The restriction $R(s)\ge s_0>0$ in the enunciation of Theorem 4, p. 296, makes the proof on p. 297 meaningless.}

Some applications involving G-functions are given.

J. L. Griffith (Kensington, Ont.)

1176:

Mauldon, J. G. An inversion formula for a generalized transform. Mathematika **4** (1957), 146–155.

Let

$$(1)\qquad \phi_n(t)=\int_{-\infty}^{\infty}(t-\lambda)^{-n}dF(\lambda),$$

where $F(\lambda)$ is a distribution function and $\int^{\lambda} \log|\lambda-\lambda_j|dF(\lambda)$ $(j=1, 2)$ converges; then the author inverts (1) by proving that

$$F(\lambda_2)-F(\lambda_1)=(2\pi i)^{-1}\int_{-i\infty}^{i\infty}\{\phi_n(\lambda_2+t)-\phi_n(\lambda_1+t)\}t^{n-1}dt$$

(the integral being taken along the imaginary axis). The method of proof is as follows. Let

$$K(t, \lambda)=t^{n-1}\{(\lambda_2-\lambda+t)^{-n}-(\lambda_1-\lambda+t)^{-n}\}$$

and let C denote a suitably chosen curve. The author first establishes

$$\int_C\Big\{(2\pi i)^{-1}\int_{-\infty}^{\infty}K(t, \lambda)dF(\lambda)\Big\}dt=\int_{-\infty}^{\infty}\Big\{(2\pi i)^{-1}\int_C K(t, \lambda)dt\Big\}dF(\lambda)$$

and then proves that the integral with respect to t is a step function whose value is one when $\lambda_1<\lambda<\lambda_2$ and zero for all other values of λ.

With two extra conditions the author also obtains the two following inversion formulae:

$$F(\lambda_1)=(2\pi i)^{-1}\lim_{R\to\infty}\int_{-R-iR_n}^{-R+iR^n}t^{n-1}\phi_n(\lambda_1+t)dt \qquad ((n-1)/n<a<1),$$

$$F(\lambda_1)=1-\alpha+(2\pi i)^{-1}\lim_{R\to\infty}\int_{Re^{-i\alpha\pi}}^{Re^{i\alpha\pi}}t^{n-1}\phi_n(\lambda_1+t)dt \qquad (0<\alpha<1).$$

The author applies these results to the problem of obtaining various sampling distributions. For example, select independently $(n-1)$ points x_i $(i=1, 2, \cdots, n-1)$ at random in the interval $(0, 1)$ and let $u_1, u_2, \cdots, u_n$ denote (in descending order of magnitude) the lengths of the n intervals into which $(0, 1)$ is divided by the x_i. Then the sampling distribution of λ, where $\lambda=\sum_{i=1}^{n}d_iu_i$, and the d_i are any given real numbers, can be obtained.

C. Fox (Montreal, P.Q.)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See also 962, 963, 964, 1232, 1423, 1544.

1177:

***Tricomi, F. G. Integral equations.** Pure and Applied Mathematics. Vol. V. Interscience Publishers, Inc., New York; Interscience Publishers Ltd., London, 1957. viii+238 pp. $7.00.

In the preface the author points out the difficulties in writing a book on integral equations which can have meaning for the engineer and physicist while still coming to grips with the mathematics involved. In the opinion of the reviewer, Tricomi has had decided success in overcoming these difficulties. In a field which suffers from a lack of readable texts this book is a welcome arrival.

The middle portions of the book, chapters two and three, contain an exposition of classical Fredholm and Hilbert-Schmidt theory. The Fredholm theorem is established (for square integrable kernels) through approximation by degenerate kernels, but a résumé of Fredholm's method, using the Hadamard determinant theorem, is also presented. Chapter three contains a rather detailed discussion of orthonormal sequences and the Hilbert-Schmidt theory for symmetric kernels, and includes a description of the Ritz method. There is a short discussion of positive kernels and Mercer's theorem is proved. Chapter three also contains a concise treatment of the application of integral equations to Sturm-Liouville theory.

The author begins and ends the book with topics which are somewhat less standard. Chapter one is a rather extensive study of Volterra equations. In addition to serving as an introduction to the methods and difficulties in the theory of integral equations, this is a subject of considerable interest in itself, which often is lost in the shadow of Fredholm theory. Some interesting applications are made to differential equations.

Chapter four contains some remarks on singular integral equations, that is, those with kernels which are not square integrable. This chapter should be of particular aid to those who, for either physical or mathematical reasons, wish to read the extensive work of Muskhelishvili et al. The ideas which are essentially function theoretic are clearly presented here without the complications of generality surrounding Muskhelishvili's book "Singular integral equations" [Gostehizdat, Moscow-Leningrad, 1946; MR 8, 586; 15, 434]. This chapter also includes some bits and pieces from the little understood field of nonlinear integral equations.

That there are some complete omissions in this book is to be expected. There is only a little mention of application of Laplace transforms in Volterra equations, and there is no attempt to describe Fourier transform methods for the so-called Wiener-Hopf equations. Also omitted is the discussion of singular integral equations in two dimensions which arise in existence theorems for differential equations in three variables. On the whole, though, the book should serve as a nearly complete reference work for those in any field of classical integral equation theory.

R. C. MacCamy (Pittsburgh, Pa.)

1178:

***Gol'dfain, I. A. On a class of linear integral equations.** American Mathematical Society Translations. Ser. 2, Vol. 10, pp. 283–290. American Mathematical Society, Providence, R. I., 1958. iv+409 pp. $6.60.

Translated from Uč. Zap. Moskov. Gos. Univ. **100,** Matematika, Tom I (1946), 104–112; MR 13, 247.

1179:

***Lyubič, Yu. I. On a class of integral equations.** American Mathematical Society Translations, Ser. 2, Vol. 10, pp. 291–310. American Mathematical Society, Providence, R. I., 1958. iv+409 pp. $6.60.

Translated from Mat. Sb. N.S. **38(80)** (1956), 183–202; MR 18, 135.

1180:

Slugin, S. N. Approximate solution of integral equations given in implicit form. Mat. Sb. N.S. **43(85)** (1957), 3–8. (Russian)

The author considers the equation

$$y(x)=\int_a^b K(x, s)F(s, y(s))ds$$

and gives an iteration procedure to find approximate solutions.

J. J. Kohn (Waltham, Mass.)

1181:

Sparenberg, J. A.; Braakman, T. C.; and Benthem, C. W. Discussion of a Wiener-Hopf type integro-differential equation. Appl. Sci. Res. B. 6 (1957), 312–322.

The equation is $f''(x)=ia\int_0^{\infty}(f(t)-ibf^{(iv)}(t))k(x-t)dt$, where a, b are >0 and $k(x)=\int_1^{\infty}(s^{-1}-s^{-3})\exp(-c|x|s)ds$,

c having positive real part. The solution is to be suitably small at infinity and smooth at $x=0$; this determines f up to a constant factor. The authors investigate the quantity $f(0)/f'(0)$ (required in a physical problem) when b is small, and obtain an approximation of the form $p_0+p_1 b\log b+p_2 b$; the coefficients p_0, p_1, p_2 involve definite integrals containing the Fourier transform of $k(x)$. The equation with $b=0$ was discussed, in connexion with the theory of the anomalous skin effect, by G. E. H. Reuter and E. H. Sondheimer [Proc. Roy. Soc. London. Ser. A. **195** (1948), 336–364].

G. E. H. Reuter (Manchester)

1182:

Čerskiĭ, Yu. I. Equations of convolution type. Izv. Akad. Nauk SSSR. Ser. Mat. **22** (1958), 361–378. (Russian)

Integral equations involving convolutions of functions are studied. Notations: $Sf=f(x)\operatorname{sgn} x$, $f_+=\frac{1}{2}(I+S)f$, $f_-=\frac{1}{2}(-I+S)f$, $\{\alpha,\beta\}$ is the set of functions such that $f_+(x)e^{-\alpha x}$, $f_-(x)e^{-\beta x}\in L^2(-\infty,\infty)$, $V_y f=F$ denotes the Fourier transform of f, existing in the complex plane if $f\in\{\alpha,\beta\}$, $\alpha\leq\beta$. A convolution is an operator A, $A\varphi=V_y^{-1}[A(\zeta)V_y\varphi]$, where $A(x+iy)$ is defined, $\varphi\in\{\alpha,\beta\}$, $A\varphi\in\{\alpha_1,\beta_1\}$ where $\alpha\leq\alpha_1\leq y\leq\beta_1\leq\beta$. In this sense B is not necessarily a convolution, where $B\varphi=k*\varphi=(2\pi)^{-1}\int_{-\infty}^{\infty}k(x-t)\varphi(t)dt$, but is the sum of two convolutions, when $k\in\{\beta,\alpha\}$. Equations of the type $A_1\varphi+A_2S\varphi=f$ and similar forms are considered; the Wiener-Hopf equation is a particular case of this form; φ is to be in a class $\{\alpha,\beta\}$. In some cases application of V gives a solution; but, if this is not the case, the equation can be put in the form of a finite sum $\sum M_k\varphi=0$ where $M_k\varphi\in\{\alpha_k,\alpha_k\}$, $\alpha_1>\alpha_2>\cdots>\alpha_n$, and V applied to each term. It is shown that $V_{\alpha_k}M_k\varphi=\Omega_k(x+i\alpha_k)-\Omega_{k-1}(x+i\alpha_{k-1})$, where $\Omega_j(\zeta)$ is a Fourier transform of a function of $\{\alpha_{j+1},\alpha_j\}$. Specific examples, and methods of resolution, are considered; the result in each case is a boundary value problem for a set of analytic functions, and under certain circumstances conditions for solubility and solutions are given. Details are too complicated for review.

J. L. B. Cooper (Cardiff)

1183:

Pokornyi, V. V. On the analyticity of solutions of certain nonlinear equations. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. no. **2** (1956), 39–45. (Russian)

1184:

Schaefer, Helmut. Neue Existenzsätze in der Theorie nichtlinearer Integralgleichungen. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **101** (1955), no. 7, 40 pp.

1185:

Smirnov, M. M. On singular solutions of non-linear integral equations. Vestnik Leningrad. Univ. Ser. Mat. Fiz. Him. **9** (1954), no. 11, 3–17. (Russian)

1186:

***Gagaev, B. M. Existence theorems for solutions of integro-differential equations.** American Mathematical Society Translations, Ser. 2, Vol. 10, pp. 311–318. American Mathematical Society, Providence, R. I., 1958. iv+409 pp. $6.60.

Translated from Uč. Zap. Kazan. Univ. **115** (1955), no. 14, 21–28; MR **18**, 301.

FUNCTIONAL ANALYSIS

See also 924, 925, 980, 998, 1026, 1061, 1114, 1119, 1139, 1166, 1172, 1296, 1349, 1535.

1187:

***Day, Mahlon M. Normed linear spaces.** Ergebnisse der Mathematik und ihrer Grenzgebiete. Neue Folge. Heft 21. Reihe: Reelle Funktionen. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. iv+139 pp. DM 28.00.

In the words of the author, "this book contains a compressed introduction to the study of normed linear spaces and to that part of the theory of linear topological spaces without which the main discussion could not well proceed." The emphasis is on the spaces themselves, their structure, their geometry, rather than on operators or the algebra of operators.

Chapter I introduces linear spaces, linear functionals, linear topological spaces, and conjugate spaces, and contains a proof of the Hahn-Banach theorem and various consequences, ending with a brief introduction to cones, wedges and partial orderings. Normed spaces are introduced in Ch. II, and some basic results such as the closed graph theorem are proved by category arguments. Various topologies are compared. In Ch. III the author discusses the various kinds of completeness and compactness in linear topological spaces, including Banach spaces with weak topologies. Eberlein's theorem on the equivalence of the various types of weak compactness is proved. Topological conditions for reflexivity are given, and a brief discussion is made of completely continuous linear operators.

The topics of unconditional convergence and (Schauder) bases are covered in Ch. IV. Ch. V begins with the important results concerning compact convex sets, the Krein-Milman theorem, etc., and contains the fixed point theorem of Schauder-Tychonoff. It concludes with a treatment of continuous function spaces, including the Banach-Stone theorem and the results of Arens, Kelley, Myers and Jerison on characterizations of continuous function spaces among Banach spaces.

In Ch. VI the concept of a vector lattice is introduced. The Stone-Weierstrass theorem is proved, and a characterization of the closed linear sublattices of spaces of continuous functions over a compact Hausdorff space due to Kakutani is given, along with related results of M. and S. Krein. Conditions are discussed under which the Hahn-Banach extension property can be generalized to functions with values in a partially ordered linear space. A characterization due to Kakutani of abstract Lebesgue spaces is given, along with other properties of such spaces. Ch. VII contains a brief discussion of "metric geometry in normed spaces," including some results on rotundity (strict convexity) and smoothness (the unit sphere is smooth if at every point of its boundary there is only one hyperplane). The chapter concludes with various characterizations of inner-product spaces, beginning with that of P. Jordan and J. von Neumann. The last chapter contains a brief but useful "reader's guide" to the literature, followed by an adequate bibliography.

This book will be useful to any who wish to do research on linear spaces, and provides an excellent and much needed text for advanced courses or seminars in this field.

D. H. Hyers (Los Angeles, Calif.)

1188:

Marinescu, Gheorghe. Différentielles de Gâteaux et Fréchet dans les espaces localement convexes. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1(49)** (1957), 77–86.

Let E and F be real locally convex spaces, T a mapping of E into F. If T is defined on a neighbourhood of $x \in E$, the existence of its Gâteaux differential at x amounts to that of the partial derivative of T in arbitrary but fixed directions issuing from x:

$$\lim_{t \to 0} \frac{T(x+th)-T(x)}{t},$$

$h \in E$ being fixed but arbitrary. For fixed x this limit, qua function of $h \in E$, is shown to be linear; it is denoted by T_x' and called the derivative of T at x (even though it has the nature of a differential rather than a derivative). Analogues are given of the Lagrange formula (first mean value theorem) and of the conditions for complete integrability. The Fréchet differential, whose existence implies that of the Gâteaux differential, amounts to the existence of a continuous linear map $L: E \to F$ such that, for any given continuous seminorm q on F, there exists a continuous seminorm p on E for which

$$q[T(x+h)-T(x)-L(h)]=o(p(h))$$

as $h \to 0$ in E. The "chain rule" for Fréchet differentials is established, and sufficient conditions are given in order that the Gâteaux differential T_x' shall at the same time be the Fréchet differential. *R. E. Edwards* (Woking)

1189:

Marinescu, G. Algèbres d'opérateurs dans un espace localement convexe. Rev. Math. Pures Appl. **2** (1957), 413–417.

A linear space E (without topology) is called the pseudo-topological union of a family $\{E_i\}_{i \in I}$ of subspaces, each equipped with a locally convex topology, if (a) $E = \bigcup_{i \in I} E_i$, and (b) for i_1 and $i_2 \in I$, there exists $i_3 \in I$ such that $E_{i_1} \cup E_{i_2} \subset E_{i_3}$, and such that the topologies in E_{i_1} and E_{i_2} are finer than the one induced by E_{i_3}. The author studies the algebra of continuous linear transformations from one locally convex space into another, and shows how it may be obtained as the pseudo-topological union of certain subspaces. *E. Michael* (Seattle, Wash.)

1190:

Pavel, Monica. Quelques propriétés des rétractes linéaires. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **5** (1956), no. 10, 19–22. (Romanian. Russian and French summaries)

The topological concepts of retract and absolute retract have obvious analogues for topological vector spaces, all mappings involved being assumed linear. The resulting concepts are "linear retract" and "absolute linear retract", and the author establishes for these a number of analogues of results known for the purely topological case. *R. E. Edwards* (Woking)

1191:

Pârvu, Monica Pavel. Sur les espaces quasi normés. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **6** (1957), no. 14, 15–23. (Romanian. French and Russian summaries)

The author considers vector spaces E with a topology defined by a "quasi-norm" $x \to \|x\| = p(x)$ which is sub-additive and satisfies the modified homogeneity condition $\|\lambda x\| = |\lambda|^r \|x\|$ for some r satisfying $0 < r \leq 1$. These are spoken of as (CN)-spaces. A few concrete examples are given. Product and quotient spaces are considered, as well as the concept of (CN)-algebra. The principal results are extensions to (CN)-spaces and algebras of well-known facts about normed spaces and algebras, the proofs being simple and direct adaptations of those current in the normed cases. *R. E. Edwards* (Woking)

1192:

Bonsall, F. F. The decomposition of continuous linear functionals into non-negative components. Proc. Univ. Durham Philos. Soc. Ser. A. **13** (1957), 6–11.

For a partially ordered locally convex topological linear space, the following two assertions are equivalent. (A) To each neighborhood G of 0 there corresponds a neighborhood H of 0 such that x belongs to G whenever there exist vectors y and z in H with $y \leq x \leq z$. (B) To each neighborhood G of 0 there corresponds a neighborhood H of 0 such that every linear functional φ satisfying $|\varphi(x)| \leq 1$ $(x \in G)$ has a decomposition $\varphi = \varphi_1 - \varphi_2$, where φ_1, φ_2 are non-negative linear functionals and $|\varphi_i(x)| \leq 1$ $(x \in H)$. This is an analogue of a theorem of Grosberg and Krein for partially ordered normed spaces [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **25** (1939), 723–726; MR **1**, 338]. The author proves also that in a space satisfying (A), a monotone sequence which converges weakly to 0 must in fact converge to 0, thus extending a theorem of Krein [ibid. **28** (1940), 13–17; MR **2**, 315].

{Remarks: (1) The generalization of the Hahn-Banach Theorem proved by the author and applied in one of his proofs is essentially the same as that given earlier by the reviewer [Duke Math. J. **18** (1951), 443–466, p. 462; MR **13**, 354]. (2) The author's principal result is closely related to theorems of Namioka [see the review below] and H. Schaefer [Math. Ann. **135** (1958), 115–141].}

V. L. Klee, Jr. (Copenhagen)

1193:

Namioka, Isaac. Partially ordered linear topological spaces. Mem. Amer. Math. Soc. no. **24** (1957), 50 pp. $1.60.

Though much space in the literature has been devoted to vector lattices, and partially ordered normed spaces have received some attention, the present paper contains the first systematic study of partially ordered linear topological spaces. (An independent and quite different approach to the topic is that of H. Schaefer, Math. Ann. **135** (1958), 115–141.) The author's treatment is well-organized and easy to read, making good use of examples and references. Unfortunately, his only reference for basic material on linear topological spaces is to the as-yet-unpublished work by J. L. Kelley, et al. [Linear topological spaces, Univ. of Kansas, hectographed, 1953, rev. ed., 1955], but much of that material is available also in Bourbaki, Eléments de mathématique, XV, XVIII, XIX [Hermann, Paris, 1953, 1955; MR **14**, 880; **17**, 1109].

The paper is divided into eight sections. Space does not permit a full discussion, but we shall describe some of the more important contents.

§ 1. Basic definitions and simple observations concerning partially ordered linear spaces.

§ 2. Extension theorems for positive linear functionals. The following is proved very simply from the Hahn-Banach theorem, and has some important corollaries. — For a linear subspace F of a partially ordered linear space E, and a linear functional f on F, the following two statements are equivalent: (i) f can be extended to a positive

linear functional on E; (ii) there is a convex set U, radial at 0, such that $f(x)\leqq 1$ whenever $x\in F$ and $x\leqq y$ for some $y\in U$.

§3. A review of relevant facts from the theory of linear topological spaces.

§4. A partially ordered linear topological space is a triple (E, C, T), where (E, C) is a partially ordered linear space and (E, T) a linear topological space. Further restrictions are often introduced, especially local convexity of the topology T and various conditions on the positive cone C, but usually the author strives for maximum generality. A set $S\subset E$ is called "full" provided it contains each order interval whose endpoints lie in S, and (E, C, T) is "locally full" provided the full T-neighborhoods of 0 form a local base. Let $F(T)$ be the vector topology for E which has as a local base at 0 the family of all sets of the form $(U+C)\cap(U-C)$, where U is a T-neighborhood of 0. The operation $F(\cdot)$ is studied, and it is proved that (E, C, T) is locally full if and only if $F(T)=T$; when T is locally convex, (E, C, T) is locally full if and only if the topology T is determined by the family of all T-continuous monotone pseudonorms.

The "order bound" topology $T_b(E, C)$ for a partially ordered space (E, C) is that which has as a local base at 0 the family of all convex circled sets which absorb all order bounded sets. The space (E, T_b) is studied, and the "order bound dual" $(E, C)^b$ is defined as the space of all linear functionals on E which are bounded on order bounded sets. In particular, $(E, T_b)^*=(E, C)^b$. Later sections study relationships among the duals $(E, T)^*$, $(E, C)^b$, $(E, C)^\star$, and $(E, C, T)^\star$, where $(E, C)^\star=C^\star-C^\star$, $(E, C, T)^\star=(E, T)^*\cap C^\star-(E, T)^*\cap C^\star$, and $C^\star$ is the dual cone consisting of all linear functionals on E which are nonnegative on C.

§5. This section is concerned with conditions on spaces E and F which insure that each positive linear map of E into F shall be continuous. The basic tool is a theorem of the reviewer [Duke Math. J. 22 (1955), 263–269; MR 16, 1030], and the principal result is as follows. Let (E, C, T) be a partially ordered metrizable linear topological space such that E is of second category and C is complete and generating. Then $T_b\subset T$. If, in addition, T is locally convex and each order bounded subset of E is T-bounded, then $T_b=T$. As a corollary there is obtained the following extension of a theorem of Nachbin [Proc. Internat. Congress Math., Cambridge, Mass., 1950, v. 1, Amer. Math. Soc., Providence, R.I., 1952, pp. 464–465]: With (E, C, T) as before, let F be a partially ordered linear space and let T_b be the order bound topology for F. Then each positive linear map of E into F is T-T_b continuous. In particular, $(E, C)^\star\subset(E, T)^*$.

§6. Though always $(E, C)^\star\subset(E, C)^b$, the reverse inclusion may fail. (In describing one such example, the author credits to W. Stenberg a type of construction which was in fact first carried out by the reviewer [op. cit., review above; p. 454].) However, there are useful conditions under which $(E, C)^b=(E, C)^\star$. One is algebraic in nature, due to F. Riesz [Ann. of Math. (2) 41 (1940), 174–206; MR 1, 147]; the others are topological and less constructive in nature. The author's main result asserts in part that for a linear functional f on (E, C, T) to be dominated by a T-continuous positive linear functional, it is necessary and sufficient that the functional p should be finite-valued on C and continuous at 0, where $p(x)=\sup\{f(u): 0\leqq u\leqq x\}$. Finiteness of p is equivalent to $f\in(E, C)^b$, and it follows that $f\in(E, C)^\star$ if and only if $f\in(E, C)^b$ and p is continuous at 0 in the largest convex topology for E. Another corollary is that $(E, C, T)^\star=(E, F(T))^*$, and thus $(E, C, T)^\star=(E, T)^*$ when (E, C, T) is locally full and locally convex. (In this connection, see also the paper by Schaefer referred to above, and the paper by Bonsall reviewed above.) There are some results concerning normality of cones (in the sense of Krein) in partially ordered pseudonormed spaces, and in addition the following results seem worthy of mention: $(E, C)^\star=(E, C)^b$ if (E, C) admits a locally convex pseudometrizable topology under which C is closed and complete. If (E, C, T) is a partially ordered Frechet space in which C is closed and generating, then $(E, C)^\star=(E, C, T)^\star=(E, C)^b$.

§7 introduces some basic notions concerning vector lattices. A set A in a vector lattice is "solid" (following G. T. Roberts [Proc. Cambridge Philos. Soc. 48 (1952), 533–546; MR 14, 395]) provided $y\in A$ whenever $|y|\leqq|x|$ for some $x\in A$. §8 begins by improving some of Roberts' results on locally solid spaces. For a lattice-ordered linear topological space, local solidness is equivalent to local fullness together with continuity of the lattice operations; moreover, when T is locally convex, local solidness is equivalent to the condition that T is determined by the continuous pseudonorms p for which $p(y)\leqq p(x)$ whenever $|y|\leqq|x|$. It is proved that if T_b is the order bound topology of a vector lattice (E, C), then T_b is the Mackey topology $m(E, (E, C)^\star)$, T_b is locally solid, and the space (E, T_b) is tonnelé if (E, C) is σ-complete. The term "ordered Banach space" is employed to denote a partially ordered Banach space (E, C, p) in which C is closed, generating, and normal in the sense of Krein. It is proved that for such a space $(E, C)^\star=(E, p)^*$; the assumption of completeness is essential. The last part of §8 sets forth conditions on an ordered Banach space which insure its being a Banach lattice. *V. L. Klee, Jr.* (Copenhagen)

1194:

Grothendieck, A. Résumé de la théorie métrique des produits tensoriels topologiques. Bol. Soc. Mat. São Paulo 8 (1953), 1–79 (1956).

This essay presents a metric theory of topological tensor products of Banach spaces. It is an outgrowth of the author's thesis [Mem. Amer. Math. Soc. No. 16 (1955); cf. also Ann. Inst. Fourier Grenoble 4 (1952), 73–112; 6 (1955–1956), 117–120; MR 15, 879, 1140; 17, 763; 18, 746] which dealt with tensor products of general locally compact spaces. Most of the questions of the general theory reduce, however, to the case of Banach spaces and the limitation of generality in the present paper is welcome. Actually the principal results of the present paper concern Hilbert space H and two classical Banach spaces of (real or complex valued) functions on a locally compact space M: the space $C=C_0(M)$ of continuous functions vanishing at infinity and the space $L=L^1(\mu)$ of functions integrable with respect to a positive Radon measure μ.

§1 (pp. 8–19) studies general $\otimes$-norms and §2 (pp. 19–40) studies the $\otimes$-norms connected with C and L. These paragraphs, as well as the first half of the next, are preparatory; almost all the results may be found in the author's thesis and hence the proofs are usually only indicated. The great amount of space devoted to the preparatory material is justified since it makes the present paper reasonably self-contained (this is all the more important in view of the differences in notation between it and the thesis).

§3 (pp. 40–57) studies the $\otimes$-norms connected with H. The main result is given in several forms, one of which asserts that every linear mapping of norm $\leqq 1$ of H into

$L^1(\mu)$ can be factored to a linear mapping of norm ≤ 1 of H into $L^2(\mu)$ and a mapping of norm $\leq \frac{1}{2}\pi$ of $L^2(\mu)$ into $L^1(\mu)$ obtained through pointwise multiplication with a suitable $f \in L^2(\mu)$. The proof of this result, as well as that of the principal theorem of the next paragraph, involves, besides the author's standard methods, a geometrical study of finite-dimensional spaces.

§ 4 (pp. 57–74) studies relations between $\otimes$-norms. The main result here is (author's nomenclature) the fundamental theorem of the metric theory of tensor products. It is presented in many equivalent forms, one of which is the following: Let u be a continuous Hermitian form on $C \times C$; then there exists μ of norm $\leq h\|u\|$ such that $u(f, \bar{f}) \leq \int f\bar{f} d\mu$ for all $f \in C$. Here h is a universal constant; its best value satisfies $\frac{1}{2}\pi \leq h \leq 2 \sinh \frac{1}{2}\pi$ (the right side may be replaced by $\sinh \frac{1}{2}\pi$ in the real case). This result allows reduction, via decompositions, of the study of continuous linear mappings between C, L, and H spaces to a study of such mappings from one H space to another. There are also other applications, e.g., to harmonic analysis and a proof of a generalized version (with an improved constant) of a theorem of J. E. Littlewood on bilinear forms [Quart. J. Math. Oxford Ser. **1** (1930), 164–174]. The paragraph concludes with some open problems concerning the structure of general Banach spaces [cf. the following review].

[For further results contained in the present paper we refer to the reviews of previous notes (without proof): C. R. Acad. Sci. Paris **239** (1954), 577–579, 607–609; Segundo symposium sobre algunos problemas matemáticos que se están estudianto en Latino América, 1954, Centro de Cooperación Científica de la UNESCO para América Latina, Montevideo, 1954, pp. 173–177; MR **16**, 596; **17**, 765.] *A. Dvoretzky* (Princeton, N.J.)

1195:

Grothendieck, A. Sur certaines classes de suites dans les espaces de Banach et le théorème de Dvoretzky-Rogers. Bol. Soc. Mat. São Paulo **8** (1953), 81–110 (1956).

The theorem mentioned in the title asserts that in every infinite-dimensional Banach space there exist unconditionally convergent series which are not absolutely convergent. It was proved by the reviewer and C. A. Rogers [(*) Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 192–197; MR **11**, 525] through a geometrical method, and another, completely different, proof was given by the author in his thesis [see the preceding review for references]. In the present paper the author applies the main geometrical lemma of (*) in order to extend the above theorem and derive some results of a metric nature on tensor products of Banach spaces.

Let l^p ($1 \leq p \leq \infty$) denote the ordinary Banach spaces of numerical sequences (a_i), and let E be an arbitrary Banach space and E' its adjoint. A sequence (x_i) in E is said to be scalarly of integrable pth power (s.i. p) if $(\langle x_i, x' \rangle) \in l^p$ for every $x' \in E'$. It is shown that for $p > 1$, $1/p + 1/p' = 1$, the space $L(l^{p'}, E)$ of continuous linear operators from $l^{p'}$ to E may be identified with the space of s.i. p sequences in E; let $M_p((x_i))$ denote the norm of the operator corresponding to the sequence (x_i). Let $l^p \check{\otimes} E$ denote the closure in $L(l^{p'}, E)$ of $l^p \otimes E$, the set of operators of finite rank (in general $l^p \check{\otimes} E$ is a proper subspace of $L(l^{p'}, E)$). The author makes a detailed study of sequences in $l^p \check{\otimes} E$. From the many results obtained we quote the following. Let E be infinite-dimensional, $1 \leq p < 2$, $1/q = 1/p - \frac{1}{2}$, $a_i \geqq 0$ and $(a_i) \in l^q$. Then there exists a sequence $(x_i) \in l^p \check{\otimes} E$ with $\|x_i\| = a_i$ ($\|\cdot\|$ is the E norm). Moreover, given any $\varepsilon > 0$ there exists such a sequence satisfying $M_p((x_i)) < \|(a_i)\|_q + \varepsilon$ ($\|\cdot\|_q$ is the l^q norm). The theorem remains valid also for $p = 2$, $q = \infty$ provided (a_i) is a null-sequence.

The paper concludes with applications to general tensor products and some conjectures.

A. Dvoretzky (Princeton, N.J.)

1196:

Sobolev, S. L. Imbedding theorems for abstract functions of sets. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 57–59. (Russian)

This discussion of continuity of some special operators continues earlier work of the author, the last preceding of which was a note in same Dokl. (N.S.) **114** (1957), 1170–1173 [MR **19**, 844]. *M. M. Day* (Urbana, Ill.)

1197:

Shmulian, Y. L. Finite-dimensional operators analytically depending on a parameter. Ukrain. Mat. Ž. **9** (1957), 195–204. (Russian. English summary)

Let $A(\zeta)$ be a linear mapping of a Banach space E into a Banach space E_1, defined and of finite rank r_ζ for ζ in a domain G of the complex plane, and suppose that $A(\zeta)$ is a holomorphic function of ζ in G.

The following results are proved. (i) There is an integer r such that $r_\zeta = r$ for all ζ in G, except perhaps for the points of an isolated set S, and $r_\zeta < r$ for ζ in S. (ii) There exist r holomorphic functions $x_1(\zeta), \cdots, x_r(\zeta)$ taking values in E_1 and linearly independent for all ζ in G, and r holomorphic functions $f_1(\zeta), \cdots, f_r(\zeta)$ taking values in the conjugate space of E, and linearly independent in $G - S$, such that

$$A(\zeta)x = \sum_{i=1}^{r} \langle f_i(\zeta), x \rangle x_i(\zeta).$$

As a corollary, it is shown that if $A(\zeta)$ is a holomorphic $n \times m$ matrix function, then the rank r_ζ is equal to a fixed number r except in an isolated set S, and one can write

$$A(\zeta) = B(\zeta)C(\zeta),$$

where $B(\zeta)$ and $C(\zeta)$ are $n \times r$ and $r \times m$ holomorphic matrix functions respectively, $C(\zeta)$ has rank r for all ζ in G, and $B(\zeta)$ has rank r in $G - S$.

The main part of the proof consists in the solution of the following problem: Given a system of functions $x_1(\zeta), \cdots, x_n(\zeta)$, taking values in a Banach space, and holomorphic in G, having maximum rank n in G, but smaller rank in a (necessarily isolated) set S, to find a system $y_1(\zeta), \cdots, y_n(\zeta)$ of holomorphic vector functions, having rank n throughout G, and linearly equivalent to the original system in $G - S$. It is shown that this problem always has a solution, and that the solution is unique up to linear equivalence with coefficients holomorphic throughout G. *F. Smithies* (Cambridge, England)

1198:

Orlicz, W.; and Pták, V. Some remarks on Saks spaces. Studia Math. **16** (1957), 56–68.

Let X be a linear space with Banach norm which is denoted by $\|\ \|$ and is called the fundamental norm. Suppose that another Banach or Fréchet norm which is denoted by $\|\ \|^*$ and is called the starred norm, is defined on X. Let X_s be the set $E_x(\|x\| \leq 1)$. The authors introduce the distance by $d(x_1, x_2) = \|x_1 - x_2\|^*$, for x_1, x_2 in X_s, and the ω-limit $x_n \xrightarrow{\omega} x_0$ by $d(x_n, x_0) \to 0$ for $x_n \in X_s$. By $X_s(\omega)$ and $X(\omega)$ they denote X_s and X with a fixed starred norm,

respectively. If $X_s(\omega)$ is complete, then it is called a Saks space. It is assumed that $\|x\| \geqq \|x\|^*$ for x in X.

Let Y be the space conjugate to the Banach space X with fundamental norm, and $Y_s(\omega)$ be the space of distributive functionals over the space X such that, for x_n, x_0 in $X_s(\omega)$, $x_n \xrightarrow{\omega} x_0$ implies $y(x_n) \to y(x_0)$. The authors prove that $Y_s(\omega)$ is closed in Y and that, given a Banach space X and a closed subspace Y_0 of Y, it is not possible to define another starred norm in such a way that the corresponding $Y_s(\omega)$ is equal to Y_0.

The authors consider the following properties of a Saks space $X_s(\omega)$: (A) if U is a distributive operation from X to a Banach space Z and if, for every linear functional η on Z, the functional $\eta(U(x))$ belongs to $Y_s(\omega)$, then U is continuous on $X_s(\omega)$; (B_2) if $y_n \in Y_s(\omega)$ such that $y_n(x) \to y(x)$ for every $x \in X$, then y_n are equicontinuous on $X_s(\omega)$; and (B_3) if U_n is a sequence of linear operations from $X_s(\omega)$ to a Banach space Z and $U_n(x) \to U(x)$ for every x in X_s then U_n is equicontinuous on $X_s(\omega)$. It is evident that (B_3) implies (B_2). The authors prove that (B_3) is a consequence of (A) and that if Z is separable, (B_2) implies (A).

The authors consider, furthermore, some other properties and their relations to the conditions Σ_1 and Σ_2. [See Orlicz, Studia Math. **11** (1950), 237–272; **15** (1955), 1–25; MR **12**, 418; **17**, 511.] They also give many examples having one of the above properties.

S. Izumi (Chicago, Ill.)

1199:

Daleckiĭ, Yu. L. On continuous rotation of subspaces in a Banach space. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 147–154. (Russian)

Let $P_k(t)$ ($k=1, \cdots, n$) be projection operators on a Banach space, defined for all t belonging to a closed interval $[a, b]$, continuous (in the operator norm) and of bounded variation in t. Suppose further, that for each t, $\sum_1^n P_k(t) = I$ and $P_j(t)P_k(t) = 0$ ($j \neq k$). Then there exists a family $U(s, t)$ of bounded operators, continuous and of bounded variation in each variable on $[a, b]$, such that $U(s, t) = U(s, r)U(r, t)$, $U(t, t) = I$, and $P_k(s)U(s, t) = U(s, t)P_k(t)$ ($k=1, \cdots, n$; $r, s, t \in [a, b]$).

M. Jerison (Princeton, N.J.)

1200:

Klee, V. L., Jr. Homogeneity of infinite-dimensional parallelotopes. Ann. of Math. (2) **66** (1957), 454–460.

The author continues his studies on homogeneity of infinite-dimensional spaces and gives proof of two theorems. He shows first that if K is an infinite-dimensional compact convex subset of a normed linear space, Z a countable closed subset of K, and μ a homeomorphism of Z into K, then μ can be extended to a homeomorphism of K onto K. The proof uses the fact that an infinite-dimensional compact convex subset of such a space is homeomorphic with the Hilbert parallelotope $P = \{x: x = (x_1, \cdots) \in l^2 \text{ and } |x_n| < 1/n\}$. Also used are two homeomorphs of P, namely, the set $Q = \{x: x \in l^2 \text{ and } \sum (nx_n)^2 \leqq 1\}$ and the unit cell C of Hilbert space $\mathfrak{H}$ under the weak topology. Six preparatory lemmas of a technical character precede the proof. The second theorem establishes the fact that there is only one isotopy class of homeomorphisms of P into P.

E. R. Lorch (New York, N.Y.)

1201:

Shibuya, Tairyū; and Iwabuchi, Chōkei. On complete orthonormal systems. Sûgaku **8** (1956/57), 30–31. (Japanese)

The authors show that every non-complete inner-product space contains a maximal orthogonal system A such that $[A]$, the least closed linear subspace including A, is not the whole space; and that if the space is separable the intersection of all such $[A]$'s is $\{0\}$.

I. Amemiya (Kingston, Ont.)

1202:

Pesonen, Erkki. Über die Spektraldarstellung quadratischer Formen in linearen Räumen mit indefiniter Metrik. Ann. Acad. Sci. Fenn. Ser. A. I. no. **227** (1956), 31 pp.

Bilinear forms $P(x, y)$, $Q(x, y)$ are considered which are defined for x and y in a linear vector set $\mathfrak{H}$ having scalar multiplication by complex numbers. The forms P and Q are linear in the first variable and conjugate symmetric. The situation of interest is that in which neither P nor Q is definite. Two cases are treated. In one of these $\mathfrak{H}$ is finite-dimensional, in the other $\mathfrak{H}$ is a separable Hilbert space. A pair of numbers μ, ν not both zero is said to be a spectral pair if there exists an element $x \neq 0$ such that for every y, $\nu P(x, y) - \mu Q(x, y) = 0$. The ratio $\lambda = \mu/\nu$, which may be infinite, is called the eigenvalue for the eigenelement x. Let $\mathfrak{P}^+$, $\mathfrak{P}^0$, $\mathfrak{P}^-$ denote the three sets in $\mathfrak{H}$ on which $P(x, x) > 0$, $P(x, x) = 0$, $P(x, x) < 0$, respectively, and let $\mathfrak{Q}^+$, $\mathfrak{Q}^0$, $\mathfrak{Q}^-$ denote the corresponding sets for Q. In the finite-dimensional case the author assumes that $\mathfrak{P}^0 \cap \mathfrak{Q}^0 = \{0\}$; he obtains the existence of a set $\lambda_1, \cdots, \lambda_r$ of distinct eigenvalues such that, if $\mathfrak{M}_j$ is the set of eigenelements corresponding to the eigenvalue λ_j, then the $\mathfrak{M}_1, \cdots, \mathfrak{M}_r$ span the n-dimensional space, and if x_i is in $\mathfrak{M}_i$ and x_j in $\mathfrak{M}_j$ then $P(x_i, x_j) = Q(x_i, x_j) = 0$ for $i \neq j$. A basic lemma in this finite dimensional case states that the assumption $\mathfrak{P}^0 \cap \mathfrak{Q}^0 = \{0\}$ implies that either 1) $\mathfrak{Q}^+ \cup \mathfrak{Q}^0 \subset \mathfrak{P}^+ \cup \{0\}$ or 2) $\mathfrak{Q}^+ \cup \mathfrak{Q}^0 \subset \mathfrak{P}^- \cup \{0\}$ or 3) $\mathfrak{Q}^- \cup \mathfrak{Q}^0 \subset \mathfrak{P}^+ \cup \{0\}$ or 4) $\mathfrak{Q}^- \cup \mathfrak{Q}^0 \subset \mathfrak{P}^- \cup \{0\}$. The author points out that this holds in a three- or higher finite-dimensional space with only real scalar multiplication. In the infinite-dimensional case, the author assumes $\mathfrak{Q}^+ \cup \mathfrak{Q}^0 \subset \mathfrak{P}^+ \cup \{0\}$ and certain boundedness conditions (both upper and lower) on the forms P and Q relative to the Hilbert space metric, and obtains a resolution of the identity E_λ such that $P(x, y) = \int_{-\infty}^{\infty} \lambda dQ(E_\lambda x, y)$.

F. J. Murray (New York, N.Y.)

1203:

Ginzburg, Yu. P. On J-contractive operator functions. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 171–173. (Russian)

Let E_+ and E_- be the projection operators on complimentary subspaces of a Hilbert space H and $J = E_+ - E_-$. Form an "indefinite scalar product" by setting $[f, g] = (Jf, g)$. A bounded operator Y is called J-contractive if $[Yf, Yf] \leqq [f, f]$ and bi-lateral J-contractive if Y and Y^* are J-contractive. There exists a one to one relation between the bi-lateral J-contractive operators Y and a subset of the contractive operators on H given by $X = (E_+Y - E_-)(E_+ - E_-Y)^{-1}$.

The operator function $Y(\zeta)$ belongs to the class K_J if (a) it is defined and holomorphic inside the unit circle with the possible exception of a countable number of points, (b) at every point of holomorphy $Y(\zeta)$ is a bi-lateral J-contractive operator, and (c) there exists a ζ_0 such that $Y^{-1}(\zeta_0)$ exists and $J - Y^*(\zeta_0)JY(\zeta_0)$ is completely continuous. If $Y(\zeta) \in K_J$, then the author announces that $Y(\zeta)$ and $Y^{-1}(\zeta)$ exist and are holomorphic with the possible exception of a set of isolated points where they have poles. The coefficients of the Laurent expansions at these poles are bounded operators.

Under more restrictive conditions on the functions $Y(\zeta)$ results are announced on convergence of operator-valued "Blaschke products" and a representation theorem

in terms of multiplicative integrals is given. The announced results generalize results of M. S. Livšic [Mat. Sb. N.S. **34(76)** (1954), 145–199; MR **16**, 48] and V. P. Potapov [Trudy Moskov. Mat. Obšč **4** (1955), 125–236; MR **17**, 958]. *A. Devinatz* (St. Louis, Mo.)

1204:

Zuhovickiĭ, S. I. Correction to the article "Some theorems of the theory of Čebyšev approximations in Hilbert space". Mat. Sb. N.S. **43(85)** (1957), 504. (Russian)

The original appeared in same Sb. **37(79)** (1955), 3–20 [MR **17**, 388]. The correction consists of slightly strengthening the conditions on the operator-valued functions $F_k(q)$.

1205:

Volkov, V. I. On the convergence of sequences of linear positive operators in the space of continuous functions of two variables. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 17–19. (Russian)

In this note some results of P. P. Korovkin [same Dokl. (N.S.) **90** (1953), 961–964; MR **15**, 236] are extended to the case of two variables. Let $L_n(f(\xi,\eta);x,y)$ denote a sequence of positive linear operators defined on the set of functions $f(x,y)$ continuous on a bounded closed region $\bar{D}$. The principal results: Th. 1: Let $L_n(f_i(\xi,\eta);x,y)\to f_i(x,y)$ uniformly on $\bar{D}$ for $f_1=1$, $f_2=x$, $f_3=y$, $f_4=x^2+y^2$. Then $L_n(f(\xi,\eta);x,y)\to f(x,y)$ uniformly on $\bar{D}$ for every f continuous on $\bar{D}$. Th. 3: There do not exist 3 functions $g_k(x,y)$, continuous on $\bar{D}$, such that $L_n(g_k(\xi,\eta);x,y)\to g_k(x,y)$ uniformly on $\bar{D}$ $(k=1,2,3)$ implies $L_n(g(\xi,\eta);x,y)\to g(x,y)$ uniformly on $\bar{D}$ for every $g(x,y)$ continuous on $\bar{D}$.

The proof of Th. 1 is given; that of Th. 3 is stated to follow from the method of Korovkin [cited above]. The author points out that these results extend to functions $f(x_1,\cdots,x_m)$ of m variables: In Th. 1 one replaces $1, x, y, x^2+y^2$ by the $m+2$ functions $1, x_1, \cdots, x_m, \sum_1^m x_i^2$; and in Th. 3 the three functions are replaced by $m+1$ functions $g_k(x_1,\cdots,x_m)$.

I. M. Sheffer (State College, Pa.)

1206:

Markuševič, L. A. On rings of continuous functions on a circumference. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 327–334. (Russian)

In the Banach algebra C of all continuous complex-valued functions on the circle $|\lambda|=1$, let $[\phi,\psi]$ be the smallest closed subalgebra containing the functions ϕ, ψ, and 1. J. Wermer [Amer. J. Math. **76** (1954), 853–859; MR **16**, 265] conjectured that if $[\phi,\psi]\neq C$, where ϕ and ψ together separate the points of the circle, then the curve $z_1=\phi(\lambda)$, $z_2=\psi(\lambda)$ in complex 2-space bounds a piece of an analytic surface. He showed that the conjecture is valid when ϕ is one-one or when $\phi(\lambda)=\lambda^2$. Here, the author verifies Wermer's conjecture for the case $\phi(\lambda)=\lambda^n$. His proof is based upon Theorem 2: If $[\lambda^n,\psi(\lambda)]\neq C$, where λ^n and $\psi(\lambda)$ separate the points of the circle, then $\psi(\lambda)$ is a root of an equation $y^n+a_1(\lambda^n)y^{n-1}+\cdots+a_n(\lambda^n)=0$, where the $a_i(z)$ are analytic for $|z|<1$ and continuous for $|z|\leq 1$.

A serious misstatement of Theorem 1 is rectified in the same journal **13** (1958), no. 1(79), 239.

M. Jerison (Princeton, N.J.)

1207:

Zaharov, V. K. Imbedding theorems for a space having its metric degenerating on a rectilinear portion of the domain boundary. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 468–471. (Russian)

[See also #1114.]

Let D be a finite region in the upper half-plane with boundary $\Gamma=\Gamma_0\cup\Gamma_1$, where Γ_0 is a segment of the real axis. On the manifold Ω^0 of all functions $u^0(x,y)$ continuous in D, having bounded piecewise continuous second derivatives and vanishing on a certain bounded strip in D, the author defines a gradient-like operator G, which associates with each u^0 its three second derivatives. In the manifold R^0 of elements Gu^0 he introduces a scalar product $\{Gu^0, Gv^0\}$, equal to an integral over D of a sum of terms of the form

$$a_{ijkl}\frac{\partial^2 u^0}{\partial x^i\partial x^j}\frac{\partial^2 v^0}{\partial x^k\partial x^l}\quad (i,j,k,l=1,2;\ x^1=x,\ x^2=y),$$

and defining a metric in R^0. $\dot{R}$ and $\dot{\Omega}$ are the respective closures of R^0 and Ω^0 with respect to this metric.

The author proves two theorems: The first states under what conditions on the coefficients a_{ijkl} and on the argument y an arbitrary function in $\dot{\Omega}$, as well as its first derivatives, has mean value zero on various parts of Γ. The second provides upper bounds for the mean square with respect to certain weighting functions of such an arbitrary function and of its first derivatives. These weighting functions have rather complicated properties.

R. N. Goss (San Diego, Calif.)

1208:

Marinescu, G. Distributions à valeurs dans le dual d'un espace de Banach. Math. Ann. **134** (1958), 195–204; correction, ibid., 476.

L'auteur introduit l'espace $\mathscr{D}'(B')$ des distributions à valeurs dans le dual B' d'un espace de Banach B, comme le dual $(\mathscr{D}(B))'$ de l'espace $\mathscr{D}(B)$ des fonctions indéfiniment différentiables à support compact à valeurs dans B. [Il y a une légère erreur dans la définition des seminormes de la topologie limite inductive de $\mathscr{D}(B)$, erreur signalée par l'auteur lui-même dans un correctif ultérieur indiqué ci-dessus.] L'auteur étudie la dérivée, la primitive, la multiplication, la solution d'équations différentielles, la transformation de Fourier (en remplaçant $\mathscr{D}(B)$ par $\mathscr{S}(B)$).

Ces résultats se trouvent en majeure partie dans Bruhat, Bull. Soc. Math. France **84** (1956), 97–205 [MR **18**, 907]; l'auteur signale à la fin de son article qu'il n'a eu connaissance du travail de Bruhat que juste avant la parution de son article. *L. Schwartz* (Paris)

1209:

O'Keeffe, Jeremiah. Singularities of Hadamard's finite part of improper integrals in the distributions of Schwartz. Rend. Circ. Mat. Palermo (2) **6** (1957), 65–82.

L'auteur définit les parties finies d'intégrales divergentes, et introduit les distributions pseudo-fonctions $Pf\ Y(x)x^m$, $Pf\ r^m$, $Pf\ Y(x)x^m\log x$, $Pf\ r^m\log r$, pour les valeurs non singulières de m seulement. Ensuite il introduit les facteurs de convergence $C(m)$, tels que le produit des distributions précédentes par $C(m)$ ne soit plus singulier aux valeurs exceptionnelles de m. Il calcule effectivement ces facteurs de convergence, et en déduit des formules relatives au calcul symbolique des opérateurs différentiels à coefficients constants usuels (laplacien, opérateur de la chaleur ou des ondes). *L. Schwartz* (Paris)

1210:

O'Keeffe, Jeremiah. Distribution theory of the operational calculus. Rend. Circ. Mat. Palermo (2) **6** (1957), 157–170.

L'auteur expose rapidement, sans démonstration, les propriétés essentielles des distributions, de la convolution, du calcul symbolique relatif à un opérateur différentiel

D à coefficients constants (groupe de distributions E^m, avec $E^m * E^n = E^{m+n}$, $E^0 = \delta$, $E^{-1} = D\delta$), et donne des exemples de calcul par transformation de Fourier pour les opérateurs différentiels usuels (laplacien, opérateur de la chaleur et des ondes); pour les démonstrations, il renvoie à un autre article. *L. Schwartz* (Paris)

1211:

Shibata, Tosio. On subspaces of the space ($\mathcal{S}$). Sûgaku **8** (1956/57), 96-100. (Japanese)

Let F be a given family of functions on R^n. The author discusses the totality of C_∞-functions $\varphi(x)$ on R^n such that every derivative of φ decreases more rapidly as $|x| \to \infty$ than any member of F, and, introducing an adequate topology on it, also discusses its dual space, and some other related spaces, generalizing the results of L. Schwartz concerning the spaces (S), (S'), (O_M) and (O_C') [Théorie des distributions, II, Hermann, Paris, 1951; MR **12**, 833; Chap. VII]. *I. Amemiya* (Kingston, Ont.)

1212:

Yamanaka, Takeoi. Une extension de la théorie des distributions de M. J. Korevaar. I. Comment. Math. Univ. St. Paul. **5** (1956), 129–136.

The author extends to R^n part of the treatment of Schwartz's distributions given for $[0, \infty)$ and $(-\infty, \infty)$ by the reviewer [Nederl. Akad. Wetensch. Proc. Ser. A **58** (1955), 368–389, 483–503, 663–674; MR **17**, 63, 354, 594; see also the reference in the following review]. A similar definition, but only for distributions of finite order on $(-\infty, \infty)$, was given by Mikusiński [Bull. Acad. Polon. Sci. Cl. III. **3** (1955), 589–591; MR **17**, 594]. Let $x = (x_1, \cdots, x_n) \in R^n$, and $p = (p_1, \cdots, p_n)$ be a set of integers $\geqq 0$. One writes $D^p f(x) = \partial^{p_1 + \cdots + p_n} f / \partial x_1^{p_1} \cdots \partial x_n^{p_n}$ and similarly defines an antiderivative of order p. A sequence $\{f_j\}$ of continuous functions on R^n is called fundamental if for every finite interval I of R^n there exist a $p \geqq 0$ and a corresponding sequence of antiderivatives of order p of the f_j which converges uniformly on I. A distribution φ on R^n is defined as a class of equivalent fundamental sequences. The derivative $D^p\varphi$ is defined by the fundamental sequence $D^p f_j$, where the f_j form a fundamental sequence of sufficiently smooth functions defining φ. It is noted that every distribution is locally equal to a finite order derivative of a continuous function, and that the present definition of a distribution is equivalent to Schwartz's. Multiplication of a distribution by an infinitely differentiable function is considered, as well as division by x_i. *J. Korevaar* (Madison, Wis.)

1213:

Yamanaka, Takesi. Une extension de la théorie des distributions de M. J. Korevaar. II. Comment. Math. Univ. St. Paul. **6** (1957), 79–88.

This second part of the paper is devoted to Fourier transforms of tempered distributions in R^n. A continuous function $F(x) = F(x_1, x_2, \cdots, x_n)$ is of slow growth if it is bounded by a polynomial in R^n, and a tempered distribution or distribution of slow growth φ is by definition a finite order distribution derivative $D^p[F]$ ($p = (p_1, p_2, \cdots, p_n)$) of such a function. The author first defines the Fourier transform of F with respect to x_1 by the formula

$$T_1F = \Big[\int_{-\delta}^{\delta} F(t, x')e^{-2\pi i t x_1} dt\Big] + \Big(\frac{\partial}{\partial x_1}\Big)^k \Big[\Big\{\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty}\Big\} F(t, x') \frac{e^{-2\pi i t x_1}}{(-2\pi i t)^k}\, dt\Big],$$

where $x' = (x_2, \cdots, x_n)$. (Thus T_1F is essentially the kth distribution derivative with respect to x_1 of the Bochner k-transform.) He then defines the Fourier transform $T_1\varphi = T_1 D^p[F]$ of φ with respect to x_1 as the distribution of slow growth given by the formula

$$T_1\varphi = (2\pi i x_1)^{p_1} D^{p'} T_1 F,$$

where $p' = (p_2, \cdots, p_n)$. With a corresponding definition for $T_k\varphi$ ($k = 2, \cdots, n$) he shows that $T_k T_m \varphi = T_m T_k \varphi$, and finally defines the Fourier transform of φ with respect to x as

$$T\varphi = T_1 T_2 \cdots T_n \varphi.$$

A similar definition is given for the conjugate Fourier transform $T^*\varphi$. The author then proves the usual properties $T^*T\varphi = TT^*\varphi = \varphi$, $TD^p\varphi = \{\prod_{k=1}^n (2\pi i x_k)^{p_k}\} T\varphi$, etc. [L. Schwartz, Théorie des distributions, T. II, Hermann, Paris, 1951; MR **12**, 833]. It may finally be remarked that the reviewer has given a treatment of Fourier transforms in R^1 quite similar to the author's (but containing other results besides) in Symposium on Harmonic Analysis and Related Integral Transforms held at Cornell University, July, 1956, Final technical report, Department of Math., Cornell Univ., Ithaca, N.Y., 1957 (mimeographed).

J. Korevaar (Madison, Wis.)

1214:

Mikusiński, J.; and Sikorski, R. The elementary theory of distributions. I. Rozprawy Mat. **12** (1957), 54 pp.

Distributions (in the sense of Schwartz and Sobolev) are defined as classes of equivalent fundamental sequences. A sequence $\{f_n\}$ of continuous functions on (a, b) is called fundamental if for some k there is a sequence $f_n^{(-k)}$ of antiderivatives of order k which converges almost uniformly on (a, b); $\{f_n\}$ is called equivalent to $\{g_n\}$ if there are sequences $\{f_n^{(-k)}\}$ and $\{g_n^{(-k)}\}$ which are almost uniformly convergent to the same function; a class of equivalent fundamental sequences is called a distribution (of finite order). This is essentially the definition given by the first author and, with considerably more detail, by the reviewer [for references see the preceding reviews].

The authors give an excellent treatment of the elementary theory of distributions of finite order, which should be of great value to physicists and engineers. This first installment deals with the following topics: derivatives of distributions, distributions as derivatives of continuous functions, convergence of sequences and series of distributions, multiplication of distributions by infinitely differentiable functions, the distribution $f(\varphi)$, where f is a distribution and φ an infinitely differentiable function, integrable functions and functions with poles as distributions, local equality of distributions, the value of a distribution at a point, existence theorems for values, the value of a distribution at infinity, the definite integral of a distribution, periodic distributions.

The treatment is very similar to that given by the reviewer, with notable improvements where it deals with the value of a distribution at a point and the concepts depending on it. Here the authors follow the work of Łojasiewicz, Wloka and Zieleźny [Bull. Acad. Polon. Sci. Cl. III. **3** (1955) 479–481; MR **17**, 594]. Thus the value of the distribution f at x_0 is defined as the limit for $\alpha \to \infty$ (if it exists in the sense of distributions) of $f(\alpha x + x_0)$; the definite integral $\int_a^b$ of the distribution f is the value at 0 (if it exists) of $\int_a^b f(x+t)dt = g(x+b) - g(x+a)$, where g is an antiderivative of f. *J. Korevaar* (Madison, Wis.)

1215:

Burkill, J. C. An integral for distributions. Proc. Cambridge Philos. Soc. **53** (1957), 821–824.

The author gives an approach to the theory of distributions which is different from that of L. Schwartz. The idea of the method is as follows. Let f and g be periodic functions of period 1; we want to define a symbol $\int_0^1 f(d^n g/dx^{n-1})$. For simplicity we take $n=3$; we shall define the above symbol by approximating sums using the generalized third derivative of g. But we want to allow for discontinuities in g. Let

$$T(x,h)=\sum_{k=1}^{n} f(x+kh)[g(x+kh)-3g\{x+(k-1)h\}+3g\{x+(k-2)h\}-g\{x+(k-3)h\}],$$

where $h=\pm 1/n$. Let $U(x,h)=\int_0^1 T(x,h)dx/h^2$; call $F(t)=\int_0^1 f(x+t)g(x)dx$, so that $U(x,h)=n[F(0)-3F(h)+3F(2h)-F(3h)]/h^2$. Then we define $\int_0^1 f(d^3g/dx^3)=-l$ if $F(t)$ has a third Riemann derivative l for $t=0$.

The author discusses integration by parts and the Fourier transform of distributions from this point of view. *L. Ehrenpreis* (Waltham, Mass.)

1216:

Stein, E. M.; and Weiss, Guido. On the interpolation of analytic families of operators acting on H^p-spaces. Tôhoku Math. J. (2) **9** (1957), 318–339.

Let $\mathfrak{P}$ be the class of polynomials $P(w)$ with complex coefficients and let

$$\|P\|_p=\sup_{0\le r<1}\left\{(2\pi)^{-1}\int_0^{2\pi}|P(re^{i\theta})|^p d\theta\right\}^{1/p} \quad (p>0).$$

Let (M,μ) be a measure space and let $L^q(M,\mu)$ denote the space of all complex-valued measurable functions f for which

$$\|f\|_q=\left\{\int_M |f|^q d\mu\right\}^{1/q} \quad (q>0)$$

is finite. Let us now assume that T_z, $0\le \mathrm{R}z\le 1$, is an analytic family of linear transformations of $\mathfrak{P}$ into $L^1(M,\mu)$ which is of admissible growth; that is, for every $g\in L^\infty(M,\mu)$ there exist constants $0<B$, $0<b<\pi$, such that

$$\left|\int_M (T_z f)g\,d\mu\right|\le B\exp\exp b|\mathrm{Im}\,z| \quad (0\le \mathrm{R}z\le 1).$$

The principal result of the paper asserts that if T_z is analytic and of admissible growth; and if p_0, p_1, q_0, q_1 are positive numbers such that for all y, $-\infty<y<\infty$, and all P in $\mathfrak{P}$,

$$\|T_{iy}P\|_{p_0}\le A_0(y)\|P\|_{q_0},\quad \|T_{1+iy}P\|_{p_1}\le A_1(y)\|P\|_{q_1},$$

where $|A_i(y)|\le C_i\exp\exp c_i|y|$, $0<C_i$, $0<c_i<\pi$ $(i=0,1)$; then for each t, $0<t<1$, $\|T_tP\|_{q_t}\le A\|P\|_{p_t}$ where $p_t^{-1}=(1-t)p_0^{-1}+tp_1^{-1}$, $q_t^{-1}=(1-t)q_0^{-1}+tq_1^{-1}$, and where A is independent of P. This generalizes an earlier result of Calderón and Zygmund [Contributions to Fourier analysis, Princeton Univ. Press, 1950, pp. 166–188; MR **12**, 255], which was in turn based upon work of Salem and Zygmund [Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 443–447; MR **10**, 247]. These papers dealt with the case in which T_z is a constant function of z.

I. I. Hirschman, Jr. (St. Louis, Mo.)

1217:

Ézrohi, I. A. Functionals in the $C_{s_1,\cdots,s_n}$ and $L^p_{s_1,\cdots,s_n}$ spaces that are annulled on generalized polynomials in many variables. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 773–776. (Russian)

The author generalizes work by Rémès [Akad. Ukrain. RSR. Inst. Mat. Zb. Prac' **3** (1940), 21–62] and himself [Mat. Sb. N.S. **38(80)** (1956), 389–416; MR **18**, 32] on linear operators vanishing on some spaces of generalized polynomials. *M. M. Day* (Urbana, Ill.)

1218:

Cotlar, Mischa; and Bruschi, Maria. On the convexity theorems of Riesz-Thorin and Marcinkiewicz. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. Serie Segunda. Rev. **5** (1956), 162–172 (1957). (Spanish summary)

Let X be a space with points x and Lebesgue measure μ. Let $L^r(\mu)=L^r(\mu;X)$ be the set of functions $f(x)$ for which

$$\|f\|_{r,\mu}=\left\{\int_X |f(x)|^r d\mu\right\}^{1/r} \quad (1\le r\le\infty)$$

exists. Let Y be a second space for which y, ν, $L^r(\nu)=L^r(\nu;Y)$, $\|f\|_{r,\nu}$ have the meaning corresponding to x, μ, $L^r(\mu)$, $\|f\|_{r,\mu}$ respectively. The paper is concerned with mappings T of a subset D of $L^r(\mu)$ into the set of ν-measurable functions $h(y)$ defined a.e. on Y, of the following properties: (i) the domain D of T is linear; (ii) for any real λ, D is closed under the operation $f\to f^{[\lambda]}$, where $f^{[\lambda]}=f$ in the set $\{x|f(x)\ge\lambda\}$ and is zero otherwise; (iii) there exists a non-negative $K=K(T)$ such that $|T(f_1+f_2)|\le K\{|T(f_1)|+|T(f_2)|\}$; (iv) $|T(\lambda f)|=|\lambda|\,|T(f)|$. Using the notation $\mu(f;\lambda)$ for the μ-measure of the subset $\{x|f(x)\ge\lambda\}$ of X and the corresponding notation for measurable functions $h(y)$ defined in Y, the authors state the following definition: for $n=1, 2, \cdots, \infty$, $h=Tf$ is said to be of quasi-n-type (r,s) if for every f in D and every positive λ

$$\lambda^s\sum_{p=1}^{n}\{p^s-(p-1)^s\}\nu(Tf;p\lambda)\le(M\|f\|_{r,\mu})^s$$

for some constant M. The smallest M value for which this inequality holds is denoted by $M_{r,s}^{(n)}$ and called the (r,s) quasi-n-norm of T.

The main result of the paper is contained in the following theorem. With the previous definitions and assumptions, let $h=Tf$ be of quasi-n-type (r_1,s_1) and also of quasi-n-type (r_2,s_2). Then Tf is of quasi-n-type (r,s) for any couple r, s for which the point with coordinates $1/r$, $1/s$ lies on the straight line segment joining the points with coordinates $1/r_1$, $1/s_1$ and $1/r_2$, $1/s_2$. Moreover, if $\delta=\inf\{s-s_1, s_2-s\}$, then there exists a constant C_K depending only on K such that

$$(1-(n+1)^{-\delta})^{1/s}M_{r,s}^{(\infty)}\le C_K(M_{r_1,s_1}^{(n)})^t(M_{r_2,s_2}^{(n)})^{1-t}.$$

The connection of this theorem with the ones mentioned in the title of the paper is as follows. T is said to be of type (r,s) if there exists a number M such that $\|Tf\|_{s,\nu}\le M\|f\|_{r,\mu}$. The smallest such M is denoted by $N_{r,s}$ and called the (r,s)-norm of T. Then the following theorem was proved by Calderón and Zygmund [Amer. J. Math. **78** (1956), 282–288; MR **18**, 586] by extending an earlier theorem of Riesz-Thorin which was proved only for linear T. Let the domain D of T be the set of all step functions. Then: a) if Tf is of type (r_1,s_1) and also of type (r_2,s_2), then T is of type (r,s) for r, s defined as in the theorem stated above; b) $N_{r,s}\le C_K N_{r_1,s_1}^{\,t}N_{r_2,s_2}^{\,1-t}$ with C_K depending only on K; c) if $K=1$, C_K can be chosen to be 1. The authors make clear that statements a), b) of this theorem are obtained as the special case $n=\infty$ of their theorem. In particular it should be noted that $N_{r,s}=M_{r,s}^{\infty}$.

In a similar way they show that the case $n=1$ reduces to a theorem by Marcinkiewicz [C. R. Acad. Sci. Paris

208 (1939), 1272–1273] which in turn is a generalization of part a) but not of part b) of the Riesz-Thorin-Calderón-Zygmund theorem stated above. Finally, the authors express the hope of studying part c) and possible generalizations of it in later papers.

E. H. Rothe (Ann Arbor, Mich.)

1219:

Kantorovič, L. V.; and Rubinštein, G. Š. On a functional space and certain extremum problems. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 1058–1061. (Russian)

Let R be a compact metric space and let Φ be the set of all real-valued countable additive set functions defined on the Borel sets of R. The concept of "economy of transition from one mass distribution in R to another of the same total mass" suggests a new metric in the space Φ which is determined in part by the metric in R; in this metric, the conjugate space of Φ becomes the space of functions satisfying a Lipschitz condition in R. A condition for the existence of an optimal transition between two distributions of the same total mass is given. It is applied to the theory of production planning.

M. M. Day (Urbana, Ill.)

1220:

Haĭmovič, A. Some applications of F. Riesz's theorem. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 763–764. (Russian)

This note discusses special cases of the representation of linear operators by kernel functions or measures; illustrations yield Green's function and resolvent functions.

M. M. Day (Urbana, Ill.)

1221:

Lidskiĭ, V. B. On the completeness of a system of eigen elements and adjoint elements of a compact operator. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 234–236. (Russian)

This note shows that if A and B are self-conjugate operators on a Hilbert space such that (Af, f) and (Bf, f) do not change sign, and B has finite trace, then the set of characteristic and adjoint vectors of $T=A+iB$ is complete in the range of T. *M. M. Day* (Urbana, Ill.)

1222:

Ghika, Al. Une propriété des échelles spectrales de certains opérateurs auto-adjoints. Com. Acad. R. P. Romîne **7** (1957), 919–922. (Romanian. Russian and French summaries)

Let H_1, H_2 be two Hilbert spaces, T a closed linear mapping of D_T ($\overline{D}_T=H_1$) into H_2, E_1 the spectral measure of $B=(T^*T)^{\frac{1}{2}}$, E_2 the spectral measure of $C=(TT^*)^{\frac{1}{2}}$, and W the unique partially isometric mapping of H_1 into H_2 with initial manifold $\overline{B(H_1)}$ and satisfying the equation $T=WB$. In this note the author states various results concerning the relations between E_1, E_2 and W [see also F. J. Murray, An introduction to linear transformations in Hilbert space, Annals of Math. Studies, no. 4, Princeton 1941; MR 3, 50; Chap. IX].

C. T. Ionescu Tulcea (New Haven, Conn.)

1223:

Meng, Ching-Hwa. A condition that a normal operator have a closed numerical range. Proc. Amer. Math. Soc. **8** (1957), 85–88.

Consider T, any bounded normal operator on a Hilbert space. Its numerical range, that is, the set of complex numbers (Tx, x) with x running over all unit vectors in the space, is known to be convex, and to be dense in the convex closure $K(T)$ of the spectrum. Theorem: The numerical range is closed if and only if every extreme point of $K(T)$ is in the point spectrum. The main point in the proof is this: If the numerical range is closed and a_0 is an extreme point of $K(T)$, then $a_0=(Tx, x)$ for some x; the author decomposes the space by the spectral projections of T corresponding to two subsets of $K(T)$, one close to a_0 and the other not having a_0 in its convex closure; he thereby proves $Tx=a_0x$.

C. Davis (Providence, R.I.)

1224:

Nikodým, Otton Martin. Contribution à la théorie des opérateurs normaux, maximaux dans l'espace de Hilbert-Hermite séparable et complet. J. Math. Pures Appl. (9) **36** (1957), 129–146.

In this paper the author indicates how the canonical form for normal operators, which he developed to describe spectral multiplicity [C. R. Acad. Sci. Paris **238** (1954), 1373–1375, 1467–1469; MR **15**, 803], also affords a natural development of the theory of normal operators.

F. H. Brownell (Seattle, Wash.)

1225:

Taylor, Angus E.; and Halberg, Charles J. A., Jr. General theorems about a bounded linear operator and its conjugate. J. Reine Angew. Math. **198** (1957), 93–111.

Let X, Y be normed linear spaces, A a linear operator defined on all of X into Y. Consider the following possible properties of A. If $R(A)=A(X)$, then certain of the following must hold: (I) $R(A)=Y$, (II) $R(A)\neq Y$ but $\overline{R(A)}=Y$, (III) $\overline{R(A)}\neq Y$; (1) A^{-1} exists and is continuous, (2) A^{-1} exists and is not continuous, (3) A^{-1} does not exist (we say A^{-1} exists if $Ax=0$ implies $x=0$ in X). Banach [Théorie des opérations linéaires, Warsaw, 1932] proved a number of theorems giving conditions under which certain of the above six properties of A [A'] imply certain other of the properties for A' [A], where A' is the adjoint of A. The operator A is of the type N_i (N=I, II, III; i=1, 2, 3) if it satisfies simultaneously properties N and i. If A is of type N_i, A' of type M_j, then the pair (A, A') is said to have state (N_i, M_j). By making use of eleven theorems and a number of examples, the authors give a complete classification of all possible states of a pair (A, A') with respect to the above six properties. The theorems for the most part, as the authors point out, have been proved by Banach and others, but the principal contribution of the authors is to provide simplified proofs and a systematic tabulation of these results so that they will be readily available. *R. E. Fullerton* (College Park, Md.)

1226:

Goldberg, V. N. Perturbation of linear operators with a purely discrete spectrum. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 643–645. (Russian)

Dans un espace de Hilbert G on considère des opérateurs $H_\varepsilon=\varepsilon V+H_0$, $\varepsilon\geq 0$; V et H_0 étant des opérateurs non bornés, de domaines respectifs D_1 et D_0. On suppose que $D_1\subset D_0$. Moyennant des conditions convenables sur V et H_0, permettant d'affirmer en particulier que le spectre de H_ε est ponctuel pour tout $\varepsilon\geq 0$, l'A. montre que les valeurs propres et fonctions propres de H_ε convergent vers celles de H_0 lorsque $\varepsilon\to 0$. Exemple type: $H_0=-\Delta$, Δ=Laplacien, sur un ouvert Ω borné de R^n, de frontière régulière (hypothèse d'ailleurs inutile), pour les conditions aux limites de Dirichlet, $V=\Delta^2$, son domaine étant également défini par les conditions de Dirichlet.

J. L. Lions (Nancy)

1227:

Allahverdiev, Dž. È. On the completeness of a system of eigen-elements and adjoined elements of non-self-adjoint operators close to normal ones. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 207–210. (Russian)

Definitions: 1. If H is a completely continuous normal operator, let the "order" of H be $\inf\{\alpha | H^\alpha$ of finite norm$\}$. 2. G is a region of the complex plane which contains the eigenvalues of λH. 3. $\mathfrak{R}_\beta$ is the class of rays emanating from the origin and bisecting angles of size 2β (vertex at origin) which have only finitely many eigenvalues of λH in their interior. 4. $\mathfrak{R}_\beta^n$ is the class of rays R emanating from the origin and such that a rotation of $(n-1)\varphi$ (φ the argument of R) brings R into $\mathfrak{R}_\beta$. 5. $\mathfrak{R}$ is ε-complete (in G) if the (intersection of G with the) interior of every angle of size less than ε meets a ray of $\mathfrak{R}$. 6. If A is a bounded operator and $A=A_0+B_0$, A_0 completely continuous, B_0 bounded, then B_0 is called the purely bounded part of A if, for any analogous decomposition $A=A_1+B_1$, $\|B_1\| \geqq \|B_0\|$. 7. If $L(\lambda)=\sum_{i=0}^{n} \lambda^i K_i$, K_i completely continuous, y is an eigenvector corresponding to $\lambda=c$ if $y=L(c)y$; $y_1, \cdots, y_k$ are adjoint to y if $y_k=L(c)y_k+\sum_{i=1}^{k}(1/i!)(\partial^i L/\partial c^i)y_{k-i}$ (where $y_0=y$). 8. A system of eigen- and adjoint elements $U_{p,q}^{(i)}$ is n-complete, if for n arbitrary elements of Hilbert space, $f_1, f_2, \cdots, f_n$, there are sequences $f_\nu^{(N)}=\sum a_{kN}^{(i)} U_{\nu-1,k}^{(i)}$, where the coefficients $a_{kN}^{(i)}$ are ν-free, which converge to f_ν, $N\to\infty$, $\nu=1, 2, \cdots, n$. 9. H is called complete if the eigenvectors $x=\lambda Hx$ ($\lambda\neq\infty$) are 1-complete (i.e., complete). Theorem: Let: H be completely continuous, normal and complete of finite order ρ; A be a bounded operator with purely bounded part B_0; A_i, $i=1, 2, \cdots, n-1$, be completely continuous. If for $\varepsilon\leqq\pi/n\rho$, the class of rays $\mathfrak{R}_\beta^n$ for $\sin\beta > \|B_0\|$ is ε-complete in G, then the system of eigenvectors and adjoint elements of the operator $\sum_{i=0}^{n-1}(\lambda H^{1/n})^i A_i+\lambda^n H$ is n-complete. The proof involves an ingenious application of the Phragmen-Lindelöf theorem.

B. R. Gelbaum (Minneapolis, Minn.)

1228:

Sahnovič, L. A. Reduction of non-selfadjoint operators to the diagonal form. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 462–465. (Russian)

Let A be an operator of class $i\Omega$ ($(1/2i)(A-A^*)$ completely continuous) and let A have a continuous spectrum. Modulo the maximal subspace on which $A=A^*$ (the "complementary subspace" of A), A is unitarily equivalent to A_1: $f\to\alpha(x)f(x)+if_a^x f(t)\beta(t)f\,dt\beta(x)$, where all functions are r-vectors of functions, $\beta(x)$ is an $r\times r$ matrix function with nonnegative elements, f is diagonal with elements ± 1 on the diagonal, and trace $\beta=\mathrm{Sp}\,\beta=1$. If $\alpha(x)$ has an inverse $\sigma(t)$ with uniformly bounded derivative $\sigma'(t)=p^2(t)$, then A_1 is similar to A_2: $f\to xf(x)+if_a^x f(t)p(t)f\,dt\beta_1(x)$, where $\beta_1(x)=p(x)\beta(\sigma(x))$. If $p^2(x)$ is BV and Lip α ($0<\alpha\leqq 1$), a further reduction yields an A_2 of the form A_3: $f\to xf(x)+ip(x)f_a^x f(t)p(t)f\,dt$ ($f=\pm 1$). The similarity B which effects this is

$$B:\varphi\to -f(2\pi)^{-1}\frac{d}{dx}\int_a^x \varphi(\sigma)\{\exp[\pi p^2(\sigma)]-\exp[-\pi p^2(\sigma)]\}^{\frac{1}{2}} \times\exp\Big[-i\int_a^x \frac{p^2(s)}{s-\sigma}f\,ds\Big]d\sigma p(x)^{-1}.$$

If $\beta_1(x)\equiv(\mathrm{Sp}\,\beta(x))\,\beta(x)$, and if $\beta(x)$ has only N eigenvalues $\lambda_i(x)]_1^N$, $\lambda_i(x)\geqq\delta>0$, and $\|\beta_1^2(x_2)-\beta_1^2(x_1)\|<K|x_2-x_1|$, then A_2 can be reduced to diagonal form (multiplication by the independent variable).

Special applications to Schrödinger equations and certain differential systems are noted.

B. R. Gelbaum (Minneapolis, Minn.)

1229:

Brodskiĭ, M. S. On a problem of I. M. Gel'fand. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 2(74), 129–132. (Russian)

The author solves the following problem, suggested by I. M. Gel'fand [Uspehi Mat. Nauk **5** (1938), 233–234]: let $f(x)$ belong to the space $H=L_2[0, a]$, and define the operator J by

$$Jf(x)=\int_0^x f(t)dt;$$

under what conditions is the set of functions $J^n f(x)$ ($n=0, 1, 2, \cdots$) fundamental in H? It is proved that a necessary and sufficient condition for this to be true is that, for $0<\alpha\leqq a$, the set of points in $(0, \alpha)$ at which $f(x)\neq 0$ should have positive measure.

The proof also yields the following more general result. Let A be a bounded linear operator in a Hilbert space H such that (i) $A-A^*$ has rank 1, its range consisting of the scalar multiples of the element e, (ii) the set $A^n e$ ($n=0, 1, 2, \cdots$) is fundamental in H, (iii) the spectrum of A consists of the point 0 only; then A has a continuous infinity of invariant subspaces, and if H_1 and H_2 are two of these, either $H_1 \subset H_2$ or $H_2 \subset H_1$.

F. Smithies (Cambridge, England)

1230:

***Mikusiński, Jan. Operatorenrechnung.** Mathematik für Naturwissenschaft und Technik, Bd. 1. VEB Deutscher Verlag der Wissenschaften, Berlin, 1957. xii+360 pp.

A translation of "Rachunek operatorów" [Monograf. Mat. v. 30, Warsaw, 1953; MR **16**, 243].

1231:

***Šmul'yan, Yu. L. Completely continuous perturbations of operators.** American Mathematical Society Translations, Ser. 2, Vol. 10, pp. 341–344. American Mathematical Society, Providence, R. I., 1958. iv+409 pp. $6.60.

Translated from Dokl. Akad. Nauk SSSR. (N.S.) **101** (1955), 35–38 [MR **16**, 933].

1232:

Butzer, P. L. Über den Grad der Approximation des Identitätsoperators durch Halbgruppen von linearen Operatoren und Anwendungen auf die Theorie der singulären Integrale. Math. Ann. **133** (1957), 410–425.

The main result is as follows. For each $l\geqq 0$, let $T(l)$ be a bounded linear operator on the Banach space X, such that: $T(0)=I$; $T(l_1+l_2)=T(l_1)T(l_2)$; $T(l)x$ is measurable, $\int_0^1 |T(l)|dl<\infty$, and str. $\lim_{s\to 0}\varepsilon^{-1}\int_0^\varepsilon T(l)x\,dl=x$, for every fixed x in X. Then if x is in the domain D_A of the infinitesimal generator A of the semi-group $T(\cdot)$,

$$|T(l)x-x|\leqq l \max_{0\leq l'\leq l}|T(l')|\,|Ax|.$$

Conversely, if X is separable and reflexive, if $T(\cdot)$ is as before except that the condition $\int_0^1|T(l)|dl<\infty$ may be weakened to the condition $\int_0^1|T(l)x|dl<\infty$, and if $|T(l)x_0-x_0|=O(l)$, then $x_0\in D_A$. Rather sharp results on approximations in $L_p(-\infty, \infty)$ and other spaces are deduced from this. Its proof makes considerable use of methods and results due to Hille [cf. Hille and Phillips, Functional analysis and semi-groups, Amer. Math. Soc.

Colloq. Publ., v. 31, rev. ed., Providence, R.I., 1957; MR 19, 664]. *I. E. Segal* (Chicago, Ill.)

1233:

Yastrebov, Yu. M. Generalized group operators. Černivec. Derž. Univ. Nauk. Zap. Ser. Fiz.-Mat. **19** (1956), no. 4, 124–148. (Ukrainian. Russian summary)

1234:

***Dixmier, Jacques. Les algèbres d'opérateurs dans l'espace hilbertien (Algèbres de von Neumann).** Cahiers scientifiques, Fascicule XXV. Gauthier-Villars, Paris, 1957. vi+367 pp. 5500 fr.

This book presents a thorough reworking and systematic development of the theory of weakly-closed *-algebras of bounded operators on a Hilbert space, which the author renames von Neumann algebras. Although only twenty years old, this has been one of the most active and fruitful areas of modern functional analysis, as the author's bibliography of nearly two hundred items attests. The present exposition divides into three large chapters: the first on the global theory, the second on the direct integral reduction theory, and a final one presenting those standard but more technical topics which do not fit into the pattern of development chosen by the author. A valuable part of the book are the numerous exercises, which present illuminating examples and sketch additional bits of theory.

The first chapter is probably the most important, since the global theory is the principal aspect of the general theory to be developed since the original von Neumann and von Neumann-Murray papers. Its goal is the global decomposition of a von Neumann algebra according to type and finiteness, and its essential tool is the theory of traces, upon which the key definitions are based. This procedure avoids the more difficult global dimension theory of projections (which is here relegated to the third chapter), but it does not yield the complete classification analysis.

The first two sections of the first chapter give the elementary definitions of the subject and the standard ways of forming new algebras from old: the reduced algebras generated by projections, direct sums of algebras and tensor products of algebras, in addition to the basic process of forming the commutator algebra $\mathcal{A}'$. The third section introduces the various topologies found to be useful in studying operator algebras — uniform, strong, weak, ultra strong and ultra weak — and investigates the ideas and improvements stemming from the original von Neumann theorem that the strong closure of a *-algebra $\mathcal{A}$ of operators is its double commutator algebra $\mathcal{A}''$. The author next treats positive linear functionals, the homomorphisms they induce, and the special functionals (normal, ultra weakly continuous) generated by elements of the underlying Hilbert space. The central theorem of this section is the complete answer to the question of to what extent a normal homomorphism between von Neumann algebras implies a spatial isomorphism of the underlying Hilbert spaces. Section five develops the elementary theory of Hilbert algebras, which can be regarded as the abstract presentation of the way in which the left and right regular representations are treated on locally compact groups. The group situation is discussed in the exercises. These algebras are of crucial importance to the present scheme of development because they provide their generated von Neumann algebras with plenty of traces, upon which, as mentioned above, the classification theory is made to depend. In section six, probably the major section of the book, the theory of traces is developed. In particular, the basic connection with Hilbert algebras is established, and the classification and decomposition according to finiteness accomplished. The next two sections effect the global analysis according to type, and the last section contains examples of the various kinds of factors.

The course of the second chapter of necessity runs closer to the original von Neumann treatment in its general outline, although there is much change in detail and improvement in exposition. After a preliminary development of the theory of the direct integral of a measurable field of Hilbert spaces and the notion of a decomposable operator, we have the notion of a diagonalizable operator (a direct integral of constant operators), and the theorem that a bounded operator is decomposable if and only if it commutes with the commutative algebra $\mathcal{Z}$ of all diagonalizable operators. The heart of the chapter is the material on decomposable algebras. Basic are the theorem (§ 3, Theorem 3) that if $\mathcal{A}$ and $\mathcal{A}'$ are both decomposable, into the families $\mathcal{A}(\zeta)$ and $\mathcal{A}'(\zeta)$ respectively, then $\mathcal{A}(\zeta)$ and $\mathcal{A}'(\zeta)$ commute for almost all ζ, and are complementary factors if $\mathcal{Z}$ is the center of $\mathcal{A}$; the theorem (§ 3, Lemma 1) that if the base space Z has a countable basis then the measurability of the field $\mathcal{A}'(\zeta)$ follows from that of $\mathcal{A}(\zeta)$ — this is the point in the theory where von Neumann had to resort to the theory of analytic sets, and the present author still finds this necessary; and the theorem (§ 6, Theorem 2) that if $\mathcal{I}$ is a commutative von Neumann algebra such that $\mathcal{I}'$ is countably generated over $\mathcal{Z}$, then the underlying Hilbert space can be decomposed as a direct integral in such a way that $\mathcal{I}$ becomes the algebra of all diagonalizable operators. These three theorems together imply the original von Neumann decomposition theorem, to the effect that a von Neumann algebra over a separable Hilbert space is (essentially) uniquely expressible with respect to its center as a direct integral of factors, and it is curious that the author does not explicitly state this corollary. The newer material contained in this chapter is mainly to be found in sections four and five, on the reduction theory of Hilbert algebras and traces, respectively.

The third, and most technical, chapter begins with the comparison theory of projections and the associated relative dimension theory, continuing the global classification theory which was not quite completed in the first chapter. It then takes up the trace mapping of a von Neumann algebra $\mathcal{A}$ into its center $\mathcal{Z}$, and discusses the factorization of a normal trace functional into one on $\mathcal{I}$ composed with a trace mapping of $\mathcal{A}$ into $\mathcal{Z}$, as well as the existence and uniqueness of the trace mapping for semi-finite and finite algebras. There follow sections treating the "approximation theorem", the coupling operator between a von Neumann algebra and its commutator algebra, and hyperfinite factors. The chapter concludes with a demonstration of the equivalence of the trace definition of finiteness with the classical definition in terms of partial isometries. This proof requires the construction of a finite trace under the latter hypothesis, and here is found the analogue of the von Neumann-Murray theorem that the natural trace on a finite factor defined for self-adjoint T by integrating λ with respect to the dimension of the spectral projections of T, is additive. The above mentioned material on central mappings and

the approximation theorem are needed for this final step in recovering the classical classification theory.

The book concludes with five short appendices presenting technical material needed in the first and second chapters. *L. H. Loomis* (Cambridge, Mass.)

1235:

de Leeuw, K. Functions on circular subsets of the space of n complex variables. Duke Math. J. **24** (1957), 415–431.

A set X in the space C^n of n complex variables is "circular" if, whenever $(x_1, \cdots, x_n)$ is in X and $|y_i|=|x_i|$, all i, then also $(y_1, \cdots, y_n)$ is in X. Let X be a bounded circular set and $A(X)$ the Banach algebra of those complex valued functions on X which are uniform limits of polynomials on X. The author first characterizes the functions in $A(X)$ in terms of the Fourier coefficients of functions they induce on the multi-torus $\{t: t \text{ in } C^n, |t_r|=1\}$. In the special case that whenever $(x_1, \cdots, x_n)$ is in X and $|y_i|\leqq|x_i|$, all i, also $(y_1, \cdots, y_n)$ is in X, he shows that $A(X)$ consists of exactly those uniformly continuous functions f on X for which the composition $f\circ g$ is analytic in the unit disk for every analytic map g of the disk into X. Let now X be compact and circular. The principal result of the paper is then the following description of the Šilov boundary of $A(X)$: Call a point y of a set Y a "multiplicative extreme point" if whenever $(y_1^2, \cdots, y_n^2)=(u_1v_1, \cdots, u_nv_n)$ for points u and v in Y, $|u_r|=|v_r|$, all r. The Šilov boundary of $A(X)$ then coincides with the closure of the set of multiplicative extreme points of the polynomially convex hull of X, the latter being $\{y: |f(y)|\leqq \sup_{x\in X}|f(x)| \text{ for all polynomials } f\}$. The proof makes use of the map: $(x_1, \cdots, x_n)\to(\log|x_1|, \cdots, \log|x_n|)$ from C^n into R^n, and of notions related to ordinary convexity in R^n. The author also uses the fact [proved by him in Trans. Amer. Math. Soc. **83** (1956), 193–204; MR **18**, 294] that for circular sets the polynomially convex hull can be defined by monomials. Finally, the author gives a "Cauchy integral formula" for functions in $A(X)$.

J. Wermer (Providence, R.I.)

1236:

Tsuji, Kazô. W^*-algebras and abstract (L)-spaces. Bull. Kyushu Inst. Tech. (Math., Nat. Sci.) no. 3 (1957), 11–13.

The author gives the following necessary and sufficient condition for a commutative B^*-algebra A (= Banach algebra with involution satisfying $\|x^*x\|=\|x\|^2$) to have a faithful *-representation as a weakly closed, self-adjoint algebra of operators on a Hilbert space: namely, A as a Banach space is the conjugate space of a Banach space B with $\|u+v\|=\|u\|+\|v\|$ for all positive elements u and v in B (i.e., $(x^*x)(u)\geqq 0$ for all x in A, and the same for v). *W. F. Stinespring* (Princeton, N.J.)

1237:

Følner, Erling. Note on groups with and without full Banach mean value. Math. Scand. **5** (1957), 5–11.

The author's principal result is that if there does not exist a (right invariant) Banach mean value defined on the space B of all bounded real-valued functions on a group G, then B is the uniform closure of the subspace generated by functions f of the form $f(x)=h(x)-h(xa)$, where $h\in B$ and $a\in G$.

L. H. Loomis (Cambridge, Mass.)

1238:

Adel'son-Vel'skiĭ, G. M.; and Šreĭder, Yu. A. The Banach mean on groups. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 6(78), 131–136. (Russian)

A Banach mean on a group G is a right invariant non-negative linear functional L on the space of all bounded real functions on G such that $L(1)=1$. The authors give a new sufficient condition that G admit a Banach mean. For a finite ECG, let l_n be the cardinal number of E^n. Theorem: If for every finite ECG and $\varepsilon>0$, $l_n=o(\varepsilon^{l_n})$, then G admits a Banach mean. A cumbersome necessary and sufficient condition for G to admit a Banach mean is stated, without proof. Other parts of the note reproduce theorems of Dixmier [Acta Sci. Math. Szeged **12** (1950), Pars A, 213–227; MR **12**, 267]. {No reference is made to the work of Følner [Math. Scand. **3** (1955), 243–254; MR **18**, 51; and the paper reviewed above] or Day [Trans. Amer. Math. Soc. **69** (1950), 276–291; MR **13**, 357] on Banach means. A recent paper by Day [Illinois J. Math. **1** (1957), 509–544; MR **19**, 1067] is also of interest.}

E. Hewitt (Seattle, Wash.)

1239:

Ehrman, Joachim B. On the unitary irreducible representations of the universal covering group of the 3+2 deSitter group. Proc. Cambridge Philos. Soc. **53** (1957), 290–303.

The irreducible representations of the universal covering group G of the 3+2 deSitter group are determined by reducing these out with respect to representations of the universal covering group of the direct product $R_3\times R_3$ of a [2] and a [3] rotation group. One specific use of the results is made in obtaining from the representations of G those of the inhomogeneous Lorentz group by a process of contraction as defined by Inonu and Wigner [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 510–524; MR **14**, 1061].

A. Salam (London)

1240:

Naĭmark, M. A. On irreducible linear representations of the complete Lorentz group. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 583–586. (Russian)

A previous paper of the author [same Dokl. **97** (1954), 969–972; MR **16**, 218] dealing with the proper Lorentz group is extended to the complete group by exhibiting the operator representing the central inversion of three-space. The key result is the enumeration of all fully irreducible representations.

A. J. Coleman (Toronto, Ont.)

1241:

Ahmedov, K. T. The analytic method of Nekrasov-Nazarov in non-linear analysis. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 135–153. (Russian)

Verfasser untersucht Gleichungen der Form (1.4) $u=\lambda f(u)$, wo λ komplex und f ein analytischer Operator in einem B-Raum ist, so daß f in folgender Form entwickelt werden kann: $f(u_0+v)=\sum_n (1/n!)\delta^n f(u_0, v)$. Hier bedeutet δ^n die Gateauxvariation. Die Ausdrücke unter dem Summenzeichen schreibt er auch als $A_{u_0}{}^n(v, \cdots, v)$. Weiter bedeutet Λ die Menge aller Lösungspaare von (1.4). Ein Punkt in Λ heißt regulär, wenn der (immer als vollstetig vorausgesetzte) Operator $A_{u_0}{}^1$ eine Inverse besitzt. Satz 1: Ist ein $(\lambda_0, u_0)\in\Lambda$ regulär, so läßt sich in einer Umgebung dieses Punktes jede analytische Lösung $(\lambda, u)\in\Lambda$ erhalten aus einer Darstellung (2.4) $u=u_0+\sum_{i=1}^{\infty}(\lambda-\lambda_0)^i u_i$ (eindeutige analytische Fortsetzung). Die weiteren Sätze befassen sich mit dem Verhalten bei nicht-regulären Punkten. Hier wird (2.4) ersetzt durch ein System von Potenzreihen (mehrdeutige analytische Fort-

setzung) oder durch Reihen nach gebrochenen Potenzen. Als wesentliches Unterscheidungsmerkmal dient dabei die Orthogonalität bzw. Nichtorthogonalität eines Eigenelementes ψ des konjugierten Operators von A'_{u_0} zu $f(u_0)$. — Ein ausführliches Literaturverzeichnis beginnt mit Liapunov und E. Schmidt und nennt aus neuerer Zeit eine Serie von Arbeiten, die an der Aserbeidschanischen Universität entstanden. *K. Zeller* (Tübingen)

1242:

Gavurin, M. K. Analytic methods of investigation of nonlinear functional transformations. Leningrad. Gos. Univ. Uč. Zap. **137**. Ser. Mat. Nauk **19** (1950), 59–154. (Russian)

1243:

***Krasnosel'skiĭ, M. A. Some problems of nonlinear analysis.** American Mathematical Society Translations, Ser. 2, Vol. 10, pp. 345–409. American Mathematical Society, Providence, R. I., 1958. iv+409 pp. $6.60.

Translated from Uspehi Mat. Nauk (N.S.) **9** (1954), no. 3(61), 57–114 [MR **17**, 769].

CALCULUS OF VARIATIONS

1244:

***Rothe, E. H. Some applications of functional analysis to the calculus of variations.** Calculus of variations and its applications. Proceedings of Symposia in Applied Mathematics, Vol. VIII, pp. 143–151. McGraw-Hill Book Co., Inc., New York-Toronto-London, for the American Mathematical Society, Providence, R. I., 1958. 153 pp. $7.50.

Suppose f is the sum of a positive definite quadratic form Q in $\{p_i\}$ with coefficients dependent on t, a linear form and a term C dependent on t and y alone. Consider $j(y)$ defined by

$$j(y)=\int_D f\,dt. \tag{1}$$

In problem (a), $y=y_1, \cdots, y_n$ and $p_i=dy_i/dt$, and D is a t interval. In problem (b), $t=t_1, \cdots, t_n$, $p_i=\partial y/\partial t_i$ and D is an open domain for the multiple integral in which dt represents the volume element. In problem (b) an L_2 norm is defined in terms of the coefficients in Q, whence, lumping the contribution of the other terms of f in $I(y)$,

$$j(y)=\tfrac{1}{2}\|y\|^2+I(y). \tag{2}$$

Since the unit sphere in L_2 is weakly compact, the semicontinuity of $j(y)$ ensures the existence of a minimum. For problem (a) an L_2 norm is again defined in terms of the coefficients in Q. The author makes a variable transformation from y to $x=x_1, \cdots, x_n$ in terms of which equation (2) is maintained with x replacing y and $i(x)=jy(x)$ replacing $j(y)$. Under mild differentiability conditions on C, the Morse index form turns out to be the scalar product $[z, k(x, z)]$, where $K(x, z)=k(x, z)-z$ is a completely continuous symmetric operator for each fixed x and variable z ($\in L_2$) and a significant property is the number of negative characteristic values of k. (The key transformation from y to x may be viewed as the replacement of y_i by $dP/dt=x_i$ where P_i is the momentum variable associated with the Lagrangian Q.)

D. G. Bourgin (Urbana, Ill.)

1245:

***Sigalov, A. G. Two-dimensional problems of the calculus of variations in nonparametric form, transformed into parametric form.** American Mathematical Society Translations, Ser. 2, Vol. 10, pp. 319–339. American Mathematical Society, Providence, R. I., 1958. iv+409 pp. $6.60.

Translated from Mat. Sb. N.S. **34(76)** (1954), 385–406; MR **16**, 598.

1246:

Krzywicki, A.; Rzewuski, J.; Zamorski, J.; and Zięba, A. Non-local problems in the calculus of variations. II. Ann. Polon. Math. **4** (1957), 30–39.

In part I [same Ann. **2** (1955), 77–96; MR **17**, 861] the authors considered the first variation for the problem of minimizing a functional of the form

$$I=\int_a^b\int_a^b L(t, t', q(t), q(t'), \dot{q}(t), \dot{q}(t'))\,dt\,dt',$$

where q is a function of a single variable. In this paper an oscillation theorem is derived for the linear integro-differential equation which is the Euler-Lagrange equation for I when the integrand L is a quadratic form, under the assumption that there is a uniquely determined two parameter family of solutions when a and b are fixed. Then the solutions are also the solutions of a second order linear differential equation, so the classical theorem of Sturm applies. Also the second variation of the integral, and sufficient conditions for a weak minimum are discussed.

{Attention should be called to a dissertation by A. R. Jacoby [Iterated integrals in the calculus of variations, Univ. Chicago, 1946 (microfilm copies obtainable from University of Chicago Libraries)], in which sufficient conditions for a semi-strong minimum for the functional I are derived. Here, "semi-strong minimum" means merely that a uniform bound on the derivatives of comparison functions $q(t)$ is assumed. The sufficiency proof given by Jacoby is an indirect one.}

L. M. Graves (Chicago, Ill.)

1247:

Faedo, Sandro. Problemi di calcolo delle variazioni connessi a una razionale utilizzazione dell'energia delle maree. Confer. Sem. Mat. Univ. Bari no. **24** (1957), 20 pp. (one plate)

After a résumé of the special case considered by F. Bertolini [Rend. Circ. Mat. Palermo (2) **5** (1956), 43–58; MR **18**, 51], the author takes up a more general case, where n turbines are available, and determines some conditions for maximizing the energy extracted from tidal flow. *L. M. Graves* (Chicago, Ill.)

1248:

Ulin, Bengt. On a conjecture of Nelder in mathematical statistics. Compositio Math. **13** (1958), 148–149.

J. M. Hammersley [Compositio Math. **10** (1952), 241–244; MR **14**, 769] studied two conjectures (a), (b) of Nelder. He proved conjecture (a) and a weakened form of conjecture (b). [For the formulation of these conjectures see the review quoted above.] The present author shows that conjecture (b) is true in its original form.

E. Lukacs (Washington, D.C.)

GEOMETRIES, EUCLIDEAN AND OTHER

See also 1283, 1438, 1605.

1249:

***Adler, Claire Fisher. Modern geometry. An integrated first course.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1958, xiv+215 pp. $6.00.

This book attempts to fill the need for a comprehensive one semester course in geometry suited for the beginner leading from the classical synthetic Euclidean approach to the abstract modern one of algebraic analysis. In discussing the different subjects (Euclidean and projective geometry, transformation theory, non-Euclidean and metric projective geometry, parabolic, hyperbolic and elliptic geometry with appropriate models as illustrations) the author emphasizes the axiomatic approach, without ever becoming pedantic or formalistic. The historic development is everywhere organically integrated in the text. The author also touches upon the relationship between Kantian philosophy and the modern mathematical solution to fundamental logical questions. Specific and careful citations from the bibliography at the end increase considerably the usefulness of the book to the advanced student. It should prove an inspiring book to the prospective teacher. *S. R. Struik* (Cambridge, Mass.)

1250:

Bilimović, Anton. Sur les paramètres géométriques. Srpska Akad. Nauka. Zb. Rad. **55**, Mat. Inst. **6** (1957), 59–68. (Serbo-Croatian. French summary)

La variabilité des figures géométriques. Les paramètres géométriques: 1. De la forme. 2. De la grandeur et 3. De la position. Exemples. Traitement des paramètres géométriques à l'étude de géométrie élémentaire.

Résumé de l'auteur

1251:

***Dubnow, J. S. Fehler in geometrischen Beweisen.** Kleine Ergänzungsreihe zu den Hochschulbüchern für Mathematik, 19. VEB Deutscher Verlag der Wissenschaften Berlin, 1958. 64 pp. DM 3.80.

Translated by Gerhard Rainike from the Russian "Oŝibki v geometričeskih dokazatel'stvah" [Gostehizdat., Moscow, 1953; MR **15**, 339].

1252:

Fiedler, Miroslav. Über qualitative Winkeleigenschaften der Simplexe. Czechoslovak Math. J. **7(82)** (1957), 463–478. (Russian summary)

Continuing earlier work [Časopis Pěst. Mat. **79** (1954), 297–320; **80** (1955), 462–476; **81** (1956), 182–223; Schr. Forschungsinst. Math. **1** (1957), 157; MR **16**, 1045; **19**, 674, 303], the author considers the $(n-1)$-dimensional faces of an n-dimensional simplex as the nodes of a complete graph whose edges then represent the (interior) angles between the faces. A study of the subgraphs formed by the edges corresponding to acute resp. right resp. obtuse angles leads to results of which the following are typical. A simplex is called acute-angled [not obtuse-angled] if all its angles are $<[\leq]\frac{1}{2}\pi$. An acute-angled [not obtuse-angled] simplex has all its k-dimensional faces $(2\leq k\leq n-1)$ acute-angled [not obtuse-angled]. A simplex with exactly n acute and $\frac{1}{2}n(n-1)$ right angles is called right-angled; an edge is called a kathete if the angle between the two opposite faces is acute. Every right-angled simplex can be embedded in a cuboid so that its vertices and kathetes are vertices and edges of the cuboid; and conversely any connected set of n pairwise non-parallel edges of a cuboid form the kathetes of a right-angled simplex. *F. A. Behrend* (Melbourne)

1253:

Grünbaum, M. Quelques propriétés du triangle. Gaz. Mat. Fiz. Ser. A. **4** (1958), 207–211. (Romanian. French and Russian summaries)

M being a point in the plane of a triangle (T), let (M) denote the triangle formed by the projections of M upon the internal (external) bisectors of the angles of (T) and let (N) denote the triangle formed by the symmetrics of M with respect to the midpoints of the sides of (M).

The author points out several relations between the triangles (M), (N) and is thus led to the result that (M), (N) are symmetric with respect to a point. {(M), (N) are both homothetic to the medial triangle of (M), the homothetic ratios being equal to -2 and 2, respectively, hence the symmetry. Cf. N. A. Court, Modern pure solid geometry, Macmillan, New York, 1935, pp. 18–19.}

Other relations of (M) and (N) are given in terms of the four tritangent centers of (T) and based on the orthocentric property of those four points.

{The two figures which accompany the text are placed in the wrong order and have their numbers interchanged.}

N. A. Court (Norman, Okla.)

1254:

Berkes, J. Einfacher Beweis und Verallgemeinerung einer Dreiecksungleichung. Elem. Math. **12** (1957), 121–123.

Let O be an arbitrary point in the interior of a triangle $A_1A_2A_3$, R_i $(i=1, 2, 3)$ the distance from O to A_i, r_i the distance from O to the side opposite A_i. Then $R_1R_2R_3\geq 8r_1r_2r_3$. An elementary proof is given here [for a non-elementary proof see e.g. Fejes Tóth, "Lagerungen in der Ebene, auf der Kugel und im Raum", Springer, Berlin, 1953, p. 33; MR **15**, 248] and the analogous theorem $R_1R_2R_3R_4\geq 81r_1r_2r_3r_4$ is proved for tetrahedra.

1255:

Bakalyaev, O. S. Some theorems and relations in the theory of solution of triangles. Zaporiz. Derž. Ped. Inst. Nauk. Zap. Fiz.-Mat. Ser. **2** (1956), 83–95. (Ukrainian)

1256:

Benz, Walter. Zur Theorie der Möbiusebenen. I. Math. Ann. **134** (1958), 237–247.

Let Σ be a set, whose elements $A, B, C, \cdots$ are called points, and certain non-empty subsets $a, b, c, \cdots$ of which are called circles. Van der Waerden and Smid proved [Math. Ann. **110** (1935), 753–776]: If Σ satisfies (M I') through three distinct points there exists exactly one circle; (M II) if $A\in a$, $B\notin a$, then there exists exactly one circle b through B touching a at A (i.e., meeting a at A only); (M III) there exist four distinct points not on one circle; and (M IV) the full theorem of Miquel; then Σ can be described analytically as an affine plane over a commutative field, augmented by a point W, the circles of Σ being the straight lines (augmented by W) and the conics $px_1^2+qx_1x_2+rx_2^2+\cdots=0$ with fixed (positive definite) quadratic part $px_1^2+qx_1x_2+rx_2^2$. The author considers sets Σ, called Möbius planes, satisfying (M II), (M III), and (M I): through three distinct points there exists at least one circle; if a, b, c are three different circles having at least two points in common then $a\cap b=b\cap c=c\cap a$. The intersection of two different circles having at least two points in common is called a

track (Fährte). The following result is obtained: if Σ is a Möbius plane, W any point of Σ, then the tracks α containing W and the circles a through W form the "points" and "lines" of an affine plane, incidence being defined by $\alpha \subset a$. — A contact bundle with base point P is a maximal family of circles any two of which touch at P. A Möbius plane is called an (F)-plane if to every circle k touching three different circles a_1, a_2, a_3 of a contact bundle there exists (at least) one track ϕ such that $k \cap a_i \subset \phi$ $(i=1, 2, 3)$. It is shown that an (F)-plane admits at most one orthogonality relation $\perp$ satisfying (O I) $a \perp b$ implies $b \perp a$; (O II*) if $a \perp b$ then $a \cap b$ is a track; (O III) if $P \in k$, $Q \neq P$, then there exists exactly one circle k' through P, Q with $k' \perp k$. It follows that orthogonality is invariant under any circle-preserving 1-1-map of an (F)-plane onto itself. — Examples of (F)-planes: let $\mathfrak{K}$ be a field (not necessarily commutative) of characteristic $\neq 2$, R_3 the projective space over $\mathfrak{K}$; let Ω be a non-degenerate quadric in $\mathfrak{R}_3$ (i.e. the set of all points $P \in \pi(P)$ where π is a polarity, but not a null-system) which is non-empty and weakly convex (i.e. Ω does not contain any straight line); then Ω is an (F)-plane if circles are defined as the sets $\Omega \cap \varepsilon$, where ε is any plane meeting Ω in at least two points.

F. A. Behrend (Melbourne)

1257:

Benz, Walter. Beziehungen zwischen Orthogonalitäts- und Anordnungseigenschaften in Kreisebenen. Math. Ann. **134** (1958), 385–402.

In addition to (O I), (O II*), (O III) [see the preceding review] the author introduces the following properties of an orthogonality relation in a Möbius plane: (O II) if $a \perp b$, then a, b have exactly two points in common; (O IV) if a, b touch and $k \perp a$, $k \perp b$, then $a \cap b \subset k$; (O V) if $a \perp k$, $a \perp k'$, $b \perp k$, $b \perp k'$, and $k \cap k' = 0$, then $a \cap b \neq 0$; (O VI) [(O VI*)] if P, Q, R are distinct, then there exists exactly [at least] one circle through R orthogonal to all circles through P, Q. Let Σ be a Möbius plane with a relation $\perp$ satisfying (O I), (O II*), (O III); then (O IV) holds if and only if Σ is an (F)-plane; (O VI*) implies (M I'); and (O IV), (O VI*) imply (O VI). — A circle plane $(\mathfrak{K}_1, \mathfrak{K}_2)$ is defined as follows: its points are the elements of a commutative quadratic extension $\mathfrak{K}_2$ of a field $\mathfrak{K}_1$, and the point ∞; a circle through three distinct points A, B, C is the locus of all X for which the cross ratio $(A, B; C, X) \in \mathfrak{K}_1$. $(\mathfrak{K}_1, \mathfrak{K}_2)$ is a Möbius plane; it is an (F)-plane if and only if $\mathfrak{K}_2$ is a separable extension of $\mathfrak{K}_1$. The cardinal of the family of different relations $\perp$ on $(\mathfrak{K}_1, \mathfrak{K}_2)$ satisfying (O I), (O II), (O III) is (a) 1 if $\mathfrak{K}_2$ is separable and char $\mathfrak{K}_1 \neq 2$, (b) 0 if $\mathfrak{K}_2$ is separable and char $\mathfrak{K}_1 = 2$, (c) $\geq$ card $\mathfrak{K}_1 \geq \aleph_0$ if $\mathfrak{K}_2$ is inseparable. Case (a) arises if and only if (O IV) holds; the unique relation $\perp$ is then defined by $a \perp b$ if a, b intersect in exactly two points S, S', and if there exist $A \in a$, $B \in b$ (different from S, S') such that $(S, S'; A, B)^2 \in \mathfrak{K}_1$; this relation satisfies (O V) if and only if $\mathfrak{K}_1$ admits a non-trivial semi-ordering [see Sperner: Arch. Math. **1** (1948), 148–153; MR **10**, 729] in which every non-negative element is a square; it satisfies (O VI) if and only if the norm of every element of $\mathfrak{K}_2$ is a square in $\mathfrak{K}_1$; it satisfies (O V) and (O VI) if and only if $\mathfrak{K}_1$ admits an ordering in which every non-negative element is a square. Finally, it is shown that a Möbius plane possesses a relation $\perp$ satisfying (O I)-(O V), (O VI*) and the restricted pencil theorem [Ewald, Math. Ann. **131** (1956), 354–371; MR **18**, 502] if and only if it is isomorphic to the Möbius plane defined by a quadric Ω in a projective space $\mathfrak{R}_3$ over a euclidean field, where Ω has at least two points and is convex (i.e., meets any straight line in at most two points); in this case (M I'), (O VI), and the full theorem of Miquel also hold.

F. A. Behrend (Melbourne)

1258:

Babiĭ, V. P. Investigation of a certain geometric locus. Uman. Ped. Inst. Nauk. Zap. **3** (1956), 93–98. (Ukrainian)

Two circles $O(r)$ and $F_2(R)$ are given. A point A on circle O is connected with the inner or outer center of similitude F_1 of both circles. F_1A intersects circle F_2 in B. The perpendicular to F_1B at A intersects F_2B (parallel to OA) at M. Then the locus of M is $\rho = r - \dfrac{a(b - r\cos\varphi)}{r - b\cos\varphi}$; here $\rho = F_2M$, $\varphi = \angle OF_2M$, $F_2O = a$, $OF_1 = b$. Several special cases are discussed, with diagrams.

D. J. Struik (Cambridge, Mass.)

1259:

Havel, Václav. On the stereometric development of the theory of conic sections. Pokroky Mat. Fys. Astr. **2** (1957), 687–697. (Czech)

An elementary study of the behaviour of conic sections with regard to central projections and certain linear mappings in space by a synthetic method.

H. Schwerdtfeger (Montreal, P.Q.)

1260:

Steinberg, Robert. On the number of sides of a Petrie polygon. Canad. J. Math. **10** (1958), 220–221.

Each of the Platonic solids has a Schläfli symbol $\{p, q\}$, where p is the number of sides of a face and q is the number of faces at a vertex; e.g., the cube is $\{4, 3\}$. A Petrie polygon is a skew h-gon whose sides are h edges of the solid, so chosen that any two consecutive sides, but no three, belong to a face; e.g., the Petrie polygon of the cube is a skew hexagon. By an ingenious argument involving circles of symmetry, the author proves that

$$h + 2 = 24/(10 - p - q).$$

This enables him to simplify a known formula, involving expressions like $6/(h+2)$, for the numerical properties of the four-dimensional polytope $\{p, q, r\}$ [Coxeter, Regular polytopes, Methuen, London, 1948; MR **10**, 261; p. 232].

H. S. M. Coxeter (Toronto, Ont.)

1261:

***Fano, Gino; e Terracini, Alessandro. Lezioni di geometria analitica e proiettiva.** G. B. Paravia & C., Torino-Milano-Padova - Firenze - Pescara - Roma - Napoli-Catania-Palermo, 1958. viii+656 pp. 3500 Lire.

It was L. Cremona who, about 1890, inaugurated in Italy that form of instruction in geometry which combines both analytic and projective geometry, and of which the present book is an outstanding example. Originally based on the teaching of G. Fano, it was first published by both authors at Turin in 1930. A second edition appeared in 1940, reprinted in 1948 [MR **10**, 318]. This third edition, somewhat revised, is due to A. Terracini. The author uses freely the analytical or synthetic method wherever it is advisable from a didactic standpoint. There are five parts. The first part deals with analytic geometry of the plane, and the second part with analytic geometry of space; both sections concentrate on linear and quadratic curves and surfaces. He inserts a sketch of the nomographic representation of equations in three variables. At the end of the second part is a section on vectors, with some information on "cursori", sliding vectors. The third part is devoted to the elements of projective geometry of the plane, and the fourth to the applications to conic sections. The last part deals with a number of properties of space

figures, such as polarity, the metrical classification of quadrics and line complexes. An appendix gives a sketch of the point of view expressed by the Erlangen Program. The text is illuminated by 210 figures.

S. R. Struik (Cambridge, Mass.)

1262:

✱**Spain, Barry. Analytical conics.** International Series of Monographs on Pure and Applied Mathematics. Vol. 3. Pergamon Press, New York-London-Paris, 1957. ix+145 pp. $5.00.

The author's aim is to provide a compact textbook on analytic geometry for the novice, readily intelligible to any student with a sound mathematical background. It develops the concepts of geometry with the classical methods of analytical geometry. Starting with straight line, circle and the three conic sections, it leads up to the general equation of the conic (with full invariant classification) pencil of conics, homographic correspondence (cross-ratio, involution), line-coordinates (with confocal conics) and generalized homogeneous coordinates. The notation is concise, the exposition clear. There are many exercises with all the solutions given in an appendix. The book should be a help to all those who look for a text in plane analytic geometry which is neither too elementary nor too advanced. *S. R. Struik* (Cambridge, Mass.)

1263:

✱**Pickert, Günter. Analytische Geometrie. 3., bearbeitete Aufl.** Mathematik und ihre Anwendungen in Physik und Technik, Reihe A, Bd. 24. Akademische Verlagsgesellschaft, Geest und Portig K.-G., Leipzig, 1958. xii+410 pp. DM 26.00.

The first edition was reviewed in MR **15**, 339; the second, very slightly changed, was published in 1955. The present edition has a few added exercises, three essentially altered passages (in the discussion of complex metric spaces, of projections and of correlations) and a completely rewritten appendix II, whereby the axioms concerning scalar multiplication are geometrized after the manner of E. Artin.

1264:

Lenz, Hanfried. Ein kurzer Weg zur analytischen Geometrie. Math.-Phys. Semesterber. **6** (1958), 57–67.

Axioms for analytic geometry based on the work of Artin, to be found, e.g., in his "Geometric algebra" [Interscience, 1957; MR **18**, 553].

1265:

✱**Salmon, George. A treatise on the analytic geometry of three dimensions.** Revised by R. A. P. Rogers. 7th ed. Vol. 1. Edited by C. H. Rowe. Chelsea Publishing Company, New York, 1958. xxiv+470 pp. $4.95.

An unchanged reproduction of the seventh edition of 1923.

1266:

Wagner, Richard. Projektive Bewegungsgruppen. I, II. Math. Ann. **134** (1958), 308–335, 336–354.

The purpose of the paper is to define the motion groups in projective n-space in a way that makes possible the characterization of all the motion groups by means of invariance properties. This necessitates a modification of the usual concept of movability which lacks symmetry.

Projective n-space, P^n, is considered over a commutative, ordered and archimedean coordinate field. The objects of the motions are "direction elements", i.e., systems of n consecutively enclosed subspaces $H^0 \subset H^1 \subset \cdots \subset H^{n-1}$ (superscripts denote dimensions). A natural topology is introduced in the space of direction elements, and a direction element $\mathfrak{s}$ is called "locally freely movable by a group B of projectivities" if it can be transformed by an element of B into any direction element in a given neighborhood of $\mathfrak{s}$. The transformations of B which leave $\mathfrak{s}$ fixed componentwise are said to form the invariance subgroup $B_{\mathfrak{s}}$. The direction element $\mathfrak{s}$ is called "regular with respect to B" if $B_{\mathfrak{s}}$ has order 2^n and exponent 2. Now, B is called a motion group if there exists at least one locally freely movable direction element which is also regular with respect to B. The main tools in the characterization of the motion groups are certain combinations of polarities: Let $\mathfrak{P}=\{\mathfrak{Q}_0, \cdots, \mathfrak{Q}_t\}$ be a system of polarities $\mathfrak{Q}_i$, such that $\mathfrak{Q}_i$ is defined in K_i and has the kernel K_{i+1}, where $i=0, \cdots, t$; $K_0=P^n$ and $K_{t+1}=0$. The author gives a definition of the transformation of subspaces by such a system of polarities and calls the transformation a "polar system". If $\mathfrak{P}$ is a polar system, a subspace M is called "$\mathfrak{P}$-nonisotropic" if $M \oplus \mathfrak{P}(M)=P^n$. To a polar system $\mathfrak{P}$ belong three groups of projectivities: 1) the group $\Pi_{\mathfrak{P}}$ of all projectivities that commute with the polar system $\mathfrak{P}$; 2) the subgroup $I_{\mathfrak{P}} \subset \Pi_{\mathfrak{P}}$ which is generated by all involutions of $\Pi_{\mathfrak{P}}$; 3) the subgroup $\Sigma_{\mathfrak{P}} \subset I_{\mathfrak{P}}$ generated by all $\mathfrak{P}$-reflections (that is, involutions leaving fixed pointwise a given $\mathfrak{P}$-nonisotropic subspace F and its image $\mathfrak{P}(F)$). The main theorem is: "For every motion group B in P^n there exists a well-defined polar system $\mathfrak{P}$ in P^n such that $\Sigma_{\mathfrak{P}} \subset B \subset I_{\mathfrak{P}}$." The finiteness of $I_{\mathfrak{P}}/\Sigma_{\mathfrak{P}}$ implies that there is only a finite number of projectively distinct types of motion groups in P^n.

R. Artzy (Haifa)

1267:

Modenov, P. S.; and Parhomenko, A. S. The projective plane and its topology. Mat. v. Škole **1958**, no. 4, 5–17.

An expository article.

1268:

Kustaanheimo, Paul. On the relation of order in finite geometries. Rend. Mat. e Appl. (5) **16** (1957), 292–296.

Axioms are given for a relation ABC, i.e., the point B lies between the points A and C, in a Euclidean geometry over a finite field $GF(p^r)$, p an odd prime. These are: 1) ABC implies that A, B, C are collinear. There exist three points A, B, C for which ABC is valid. 2) If ABC is valid and AA', BB', CC' are parallel and A', B', C' are collinear, then $A'B'C'$ is valid. 3) If A, B, C, D are four distinct collinear points, then exactly none or exactly two of the propositions ADB, BDC, CDA are valid. It is shown that if A, B, C are three collinear points with line coordinates z_A, z_B, z_C, then ABC is valid if and only if $(z_A-z_B)/(z_C-z_B)$ is a non-square.

Marshall Hall, Jr. (Columbus, Ohio)

1269:

Kustaanheimo, Paul. On the relation of congruence in finite geometries. Rend. Mat. e Appl. (5) **16** (1957), 286–291.

A relation of congruence is introduced in finite Euclidean planes over fields $GF(p^r)$, p an odd prime. This is a relation $AB=CD$ for "segments" AB and CD. The relation is required to be an equivalence. Parallel segments are congruent if and only if they form two sides of a parallelogram. If A, B, C, D are collinear and A', B', C', D' are also collinear, and if AA', BB', CC', DD' are parallel, then the equality of AB and $A'B'$ imply the equality of CD and $C'D'$. If A, B, C are collinear and

A', B', C' are collinear, then $AB=A'B'$, $AC=A'C'$, $BC=B'C'$, $AD=A'D'$, and $BD=B'D'$ imply $CD=C'D'$. A final axiom requires that on a given line there should be a segment equal in length to one of two specified segments AB or CD. It is shown that with coordinates properly chosen, two segments are equal if and only if there are segments parallel to the given segments with one endpoint at the origin and the other endpoints both on the curve $x^2-ky^2=1$ or both on $x^2-ky^2=k$, k being a fixed non-square.

Marshall Hall, Jr. (Columbus, Ohio)

1270:

Libois, Paul. Mouvements dans les univers de Minkowski et de De Sitter. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1(49)** (1957), 59–67.

This paper contains a general description of the properties of the 4-dimensional manifolds of Minkowski and de Sitter from the point of view of abstract geometry, and a comparison with those of 3-dimensional Euclidean space. A classification of the types of motions in these manifolds is given. *G. C. McVittie* (Urbana, Ill.)

1271:

***Nordon, A. P. Elementare Einführung in die Lobatschewskische Geometrie.** Hochschulbücher für Mathematik, Bd. 35. VEB Deutscher Verlag der Wissenschaften, Berlin, 1958. ix+259 pp. DM 14.00.

A translation by Waldemar Dege and Klaus-Dieter Rieck from the Russian (not reviewed in MR) published by Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953.

1272:

Hazanov, M. B. On the application of a new theory of areas in the Lobačevskiĭ plane to deriving the curvature formula. Kabardin. Gos. Ped. Inst. Uč. Zap. **8** (1955), 21–24. (Russian)

1273:

Sasayama, Hiroyoshi. On generalized non-euclidean spaces. Comment. Math. Univ. St. Paul. **6** (1958), 141–156.

"The principal purpose of the present paper is to generalize Z. Okamura's 3-dimensional quasi non-euclidean space with a cubic surface as absolute [J. of Shiga Prefectual Junior College Ser. A **1** (1951), 11–38, 114–121] to the n-dimensional space with a hypersurface of pth order or class as absolute, corresponding to the n-dimensional quasi euclidean space [H. Sasayama, Comment. Math. Univ. St. Paul. **5** (1956), 95–114; MR **18**, 816] where $2(n+1)^2>\binom{n+p}{p}$." (From the author's summary.)

L. A. Santaló (Buenos Aires)

CONVEX SETS AND DISTANCE GEOMETRIES

See also 899.

1274:

Červená, Alena. Berichtigung zur "Bemerkung über die Lösungsfrage eines speziellen Systems von Ungleichungen durch positive Zahlen". Časopis Pěst. Mat. **83** (1958), 97–98. (Czech. Russian and German summaries)

Da die Bedingungen a), b) im Satze dieser Arbeit [dieselbe Časopis **82** (1957), 335–341; MR **19**, 462] zwar notwendig, doch nicht hinreichend sind, wird eine schärfere Bedingung a') gegeben. Es zeigt sich, dass die ursprüngliche Bedingung b) eine Folgerung dieser Bedingung a') ist.

Aus der Zusammenfassung des Autors

1275:

Nefed'ev, G. N. On equivalence of certain theorems about systems of linear inequalities. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 187–192. (Russian)

Four forms (for nonhomogeneous nonstrict systems) of the transposition theorem, credited to Minkowski, Voronoĭ, Aleksandrov and Černikov, are derived from each other. *T. S. Motzkin* (Los Angeles, Calif.)

1276:

***Scarf, H. E. On differential games with survival payoff.** Contributions to the theory of games, vol. 3, pp. 393–405. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N. J., 1957. $5.00.

Let R be a bounded region in euclidean space, with boundary B. The game may be roughly described as follows. Its position x moves, starting from some $x_0 \in R$, according to a differential equation $\dot{x}=g(x; y, z)$, where y and z are "navigation variables" controlled respectively by the opposing players I and II. The payoff to I is $b(x_1)$, where x_1 is the first point of B reached and b is a function given on B. The game is termed unbiased if, for each x and c, the scalar product $(c, g(x; y, z))$ considered as a game over (y, z) space has value 0. A direct treatment seems quite difficult, and the author instead considers a sequence of corresponding time-discrete games with time increment δ tending to 0. He introduces an appropriate second order partial differential operator L such that, under certain assumptions, if the boundary value problem $L(W)=0$, $W(x)=b(x)$ on B has a solution $W_0(x)$, then the upper and lower values of the time-discrete games tend to $W_0(x)$ as $\delta \to 0$. The assumptions include unbiasedness and the quite undesirable one that $W_0(x)$ be thrice differentiable. The result is nonetheless striking, particularly since in both R. Isaacs' pioneering work [summarized in RAND Corp. Reports RM-1391, 1399, 1411, 1486, Santa Monica, Calif., 1954 and 1955] and the reviewer's work [≠ 1277 below] the corresponding operator turns out to be of first, not second, order.

W. H. Fleming (Providence, R.I.)

1277:

***Fleming, W. H. A note on differential games of prescribed duration.** Contributions to the theory of games, vol. 3, pp. 407–412. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N. J., 1957. $5.00.

The author sets up a model of a continuous time game. Since about half of the paper is taken up with definitions, it seems inappropriate to reproduce details here. Roughly speaking, the game is of this form: At each time $t<T$ the game is in some "position" x. If now the players choose strategies y and z, respectively, the game will move to a new position x' where x' is a known function of x, y, and z. If choices of strategy are made only at a finite set of specified times $t_1, t_2, \cdots, t_n$, then we have a time-discrete game after a suitable payoff has been defined, and the usual existence theorems of game theory are applicable. In the time-continuous version the players choose times $t_1', \cdots, t_\alpha' \leq T$ and $t_1'', \cdots, t_\beta'' \leq T$, respectively, and at each of these $\alpha+\beta$ instants they will have to make choices of strategies. Since a player does not know in advance which instants his opponent will choose, he must be

prepared in advance to choose his strategy for any time t which may occur; hence the term "continuous."

It is shown that if a certain combined partial differential and functional equation has a continuously differentiable solution, then this solution will, under appropriate conditions, give the value of the time-continuous game.

D. Gale (Santa Monica, Calif.)

1278:

Buck, R. C. Preferred optimal strategies. Proc. Amer. Math. Soc. **9** (1958), 312–314.

If a player does not have a unique optimal strategy, he has an infinite number of optimal strategies. By choosing any one of them he can expect to receive at least v, the value of the game, regardless of the strategy used by his opponent. However, if the latter does not play optimally, a player may get more than v. The author gives a procedure for choosing, from the infinite set of optimal strategies, a preferred optimal strategy which takes maximum advantage of any departure from optimal play on the part of his opponent.

The procedure consists of first replacing a player's pure strategies by the extreme points of his set of optimal strategies. Then it is assumed that the opponent uses only those pure strategies which yield more than v for at least one optimal strategy of the player. It is apparent that the player's set of optimal strategies in this new game is contained in the original set. If this set is not a unique optimal strategy, the procedure is repeated. The author shows that this process either yields a unique optimal strategy or every element of the derived payoff matrix is identical.

This method for choosing a preferred optimal strategy is the same one described by George W. Brown in "A method for choosing among optimum strategies" [RM-376, The Rand Corporation, Santa Monica, Calif., 1950].

M. Dresher (Pacific Palisades, Calif.)

1279:

Zalgaller, V. A.; and Rešetnyak, Yu. G. On rectifiable curves, additive vector-functions, and displacement of segments. Vestnik Leningrad. Univ. Ser. Mat. Fiz. Him. **9** (1954), no 2, 45–67. (Russian).

1280:

Eggleston, H. G. Approximation to plane convex curves. I. Dowker-type theorems. Proc. London Math. Soc. (3) **7** (1957), 351–377.

C is the boundary of a plane bounded convex domain X while P_n is a closed polygon with n sides. If $\tau_A(X, P_n)$ is the area of the set of points within exactly one of the sets X or the interior of P_n, $T_A(X; n)$ is defined to be the greatest lower bound of $\tau_A(X, P_n)$ where X is fixed but P_n runs through all polygons with n sides. The greater part of the paper is concerned with the proof of the result $T_A(X; n-1)+T_A(X; n+1) \geqq 2T_A(X; n)$. This is analogous to a result of Dowker [Bull. Amer. Math. Soc. **50** (1944), 120–122; MR **5**, 153] for the best possible approximation to C by polygons P_n which are circumscribed about C. The proof depends on an approximation to C by a union of closed polygons for which the total number of sides of all polygons of the union does not exceed $2n$. The most important step is to show that the best possible approximation of this type, the existence of which is proved by use of the Blaschke selection theorem, consists of a union of two polygons of n sides which are not necessarily distinct. The proof, though long, is elementary.

Let $\tau_P(X, P_n)$ be the difference between the perimeter of the union of X and points interior to P_n and the perimeter of the intersection of these two sets. $T_P(X; n; i)$ [$T_P(X; n; c)$] is defined to be the greatest lower bound of $\tau_P(X, P_n)$ where P_n runs through the polygons inscribed in [circumscribed about] C. An adaptation of Dowker's methods shows that $T_P(X; n-1; i)+T_P(X; n+1; i) \geqq 2T_P(X; n; i)$, and $T_P(X; n-1; c)+T_P(X; n+1; c) \geqq 2T_P(X; n; c)$. The author points out in a footnote that these two results were obtained earlier by Molnár [Mat. Lapok **6** (1955), 210–218; MR **17**, 1235]. If $T_P(X; n)$ is the greatest lower bound of $\tau_P(X, P_n)$ for fixed X, it is shown that $T_P(X; n)=T_P(X; n; i)$.

If $T_D(X; n; i)$, $T_D(X; n; c)$, $T_D(X; n)$, are analogous numbers defined as greatest lower bounds of the distance deviation $\tau_D(X; P_n)$ between the points of X and those interior to P_n, it is shown that none of these three numbers is a convex integer function of n, where X is the regular polygon with $2r$ sides.

All the results of this paper are answers to problems proposed by Fejes Tóth [Lagerungen in der Ebene, auf der Kugel und im Raum, Springer, Berlin, 1953; MR **15**, 248].

D. Derry (Vancouver, B.C.)

1281:

Eggleston, H. G. Notes on Minkowski geometry. I. Relations between the circumradius, diameter, inradius and minimal width of a convex set. J. London Math. Soc. **33** (1958), 76–81.

Let R, r, D, d denote respectively the circumradius, inradius, diameter and minimal width of a convex set X in an n-dimensional Minkowski space M_n. If the class of Minkowski spaces of dimension n is denoted by L, the author proves that (for $M_n \in L$ and $X \subset M_n$) the following results hold good: a) $\sup \sup(R/D)=n/(n+1)$ [result due to Bohnenblust, Ann. of Math. (2) **39** (1938), 301–308]; b) $\sup \sup(d/r)=n+1$; c) $\inf \sup(R/D)=\frac{1}{2}$ (equality if and only if the indicatrix of M_n is a parallelopiped). The exact value of $\inf \sup(d/r)$ remains open; for $n=2$, the author proves that $\inf \sup(d/r)=3$.

L. A. Santaló (Buenos Aires)

1282:

Gusak, A. A. Application of the straight-line method of Čebyšev to the problem of the best construction of the Watt parallelogram. Vesci Akad. Navuk BSSR. Ser. Fiz.-Tèhn. Navuk **1957**, no. 4, 73–83. (Byelorussian. Russian summary)

1283:

Danzer, Ludwig; Laugwitz, Detlef; und Lenz, Hanfried. Über das Löwnersche Ellipsoid und sein Analogon unter den einem Eikörper einbeschriebenen Ellipsoiden. Arch. Math. **8** (1957), 214–219.

Given a bounded set M in the n-dimensional real affine space A^n whose convex closure $\bar{M}$ contains interior points, there is exactly one ellipsoid of minimal volume containing M and exactly one of maximal volume contained in the closure of $\bar{M}$. For A^2 this theorem was proved by F. Behrend [Math. Ann. **115** (1938), 397–411]. The two ellipsoids have the same center if $\bar{M}$ possesses a group of affinities on itself which does not leave a boundary point of $\bar{M}$ invariant.

Many more or less known theorems follow easily from the uniqueness of either one of the two ellipsoids, for example: A group of affinities of an affine space over the reals, the complex numbers, or the quaternions which carries a bounded open set in itself has a fixed point and leaves a hermitian form invariant. A group G of projectivities of a real projective space is a group of motions

of an elliptic metric if for any sequence $T_i \in G$ $(i=1, 2, \cdots)$ and two points a, b the relation $\lim aT_i = \lim bT_i$ implies $a=b$. A closed bounded set in A^n which admits a transitive group of affinities in itself is a subset of (the surface of) an ellipsoid. *H. Busemann* (Cambridge, Mass.)

1284:

Bambah, R. P. Maximal covering domains. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 540–543.

Let K, K' be closed bounded symmetrical convex domains. Let $c(K)$ denote the supremum of the discriminants of those lattices Λ for which $K+\Lambda$ is the whole plane. K is called maximal if $K \subset K' \neq K$ implies $c(K) < c(K')$. For every point p on the boundary of K let $t(p)$ denote the area of the largest triangle with the vertex p and inscribed in K. If $t(p)$ is independent of p, K is maximal; in particular, parallelograms and ellipses are maximal; conversely if K is maximal and strictly convex, $t(p)$ is independent of p. The author conjectures that K then is an ellipse. *P. Scherk* (Saskatoon, Sask.)

GENERAL TOPOLOGY, POINT SET THEORY

See also 948, 1190, 1200, 1315, 1320.

1285:

Hunter, Robert P.; and Swingle, Paul M. Indecomposable trajectories. Tôhoku Math. J. (2)**10** (1958), 3–10.

The authors are concerned with the topological properties which the path of a moving particle may have, path usually meaning a continuous, one-to-one image of the non-negative real line in phase space. The theorems are about paths which are dense in a connected open set. In two dimensions such a path must be an indecomposable connected set, but for higher dimensions it may also be decomposable. Existence of all possibilities is proved, and other types of connectedness are discussed for one path and also for a finite number of disjoint paths.

H. H. Corson (Seattle, Wash.)

1286:

Rudin, Mary Ellen. A topological characterization of sets of real numbers. Pacific J. Math. **7** (1957), 1185–1186.

The author designates a space E to be of class L provided E is a separable metric space for which: (1) each component of E is a point or an arc (closed, open, or half-open), and no interior point of an arc-component A is a limit point of $E-A$; (2′) each point of E has arbitrarily small neighborhoods whose boundaries are finite sets. She then proves that a space is homeomorphic with a set of real numbers if and only if the space is of class L. This proof consists of demonstrating the following: If X is a component of the space E of class L and ε is a positive number, there is a set $U(X, \varepsilon)$ which is both open and closed, contains X, and is contained in the union of X with the ε-neighborhoods of its endpoints (if any). As a direct consequence any space of class L satisfies the criterion of L. W. Cohen [Fund. Math. **14** (1929), 281–303] that a separable metric space be homeomorphic with a subset of the linear continuum. This criterion consists of (1) above, together with the two following conditions: (2) E is zero-dimensional at each of its point-components; (3) if p is an endpoint of an arc-component A, then the space $(E-A) \cup \{p\}$ is zero-dimensional at p.

W. W. S. Claytor (Washington, D.C.)

1287:

Strohl, G. Ralph, Jr. Peano spaces which are either strongly cyclic or two-cyclic. Trans. Amer. Math. Soc. **86** (1957), 297–308.

Suppose that M is a Peano space, p a point of M, and K a countable sequence of distinct points of M which converges to the point p. The space M is said to be strongly arcwise connected at p provided there is in M a simple arc containing infinitely many distinct points of K, strongly cyclic at p provided there is in M a simple closed curve containing infinitely many distinct points of K, and two-cyclic at p provided there are in M two arcs L_1 and L_2 having precisely the point p in common and each of which contains infinitely many distinct points of K. The space M is said to be strongly arcwise connected, strongly cyclic, or two-cyclic if and only if it has the corresponding property at each of its points. Strongly arcwise connected spaces were defined and studied by the reviewer and W. T. Puckett, Jr. [Amer. J. Math. **63** (1941), 554–562; Bull. Amer. Math. Soc. **47** (1941), 468–475; MR **3**, 58; **2**; 325].

The author obtains the following results for a cyclic Peano space M. (A) M is strongly cyclic if and only if given any infinite collection of open sets in M there exists a simple closed curve of M intersecting infinitely many sets of this collection. (B) Let p be a point of M, and suppose that M is strongly cyclic at p, but not two-cyclic at p. Then either p belongs to a free arc of M, or given $\varepsilon > 0$ there exists in M an arc L of diameter less than ε and an infinite collection of distinct components of $M-L$ converging to p. (C) Let p be a point of M, and suppose that M is strongly arcwise connected at p. Then in order that M fail to be strongly cyclic at p, it is necessary and sufficient that there exist in M a closed set D containing p and a collection $\{V_i\}$ of distinct components of $M-D$ having the following properties: (i) $V_i \to p$; (ii) for each i, $F(V_i) = c_i \cup d_i$, where c_i and d_i are distinct points of $D-p$; (iii) no simple closed curve in M intersects infinitely many of the regions V_i.

D. W. Hall (Endicott, N.Y.)

1288:

Weston, J. D. On the comparison of topologies. J. London Math. Soc. **32** (1957), 342–354.

This paper contains a number of topological generalizations of results from the theory of linear spaces and the theory of uniform spaces. The theorems appear naturally and simply in the course of an investigation of the relation between two topologies, T_1 and T_2, for a set X. The topology T_1 is "coupled" to T_2 if and only if every T_1-neighborhood of a point is T_2-dense in some T_2-neighborhood of the point; we write $T_1 \leqq T_2$ if and only if $T_1 \subset T_2$ and T_1 is coupled to T_2. The relation $\leqq$ is transitive, and T_1 is coupled to T_2 if and only if $T_1 \leqq T_1 \vee T_2$. If $T_1 \leqq T_2$ and (X, T_2) is of the second category, then (X, T_1) is also of the second category. If $T_1 \leqq T_2$, T_1 is a Hausdorff topology, and if (X, T_2) is an absolute G_δ (that is, can be metrized with a metric ρ so that (X, ρ) is complete), then $T_1 = T_2$. The topology T_1 is "consistent" with T_2 if the diagonal in $X \times X$ is closed relative to the product topology $T_1 \times T_2$. If T_1 is consistent with T_2 and is coupled with T_2, and if (X, T_2) is either locally compact or an absolute G_δ, then $T_1 \supset T_2$.

These results have a number of applications. In particular, the Banach-Steinhaus theorem, the closed graph theorem, and Klee's result on invariant metrics are obtained. *J. L. Kelley* (Berkeley, Calif.)

1289:
Császár, Ákos. Sur une classe de structures topologiques générales. Rev. Math. Pures Appl. **2** (1957), 399–407.

The author introduces axioms more general than the axioms for topological, uniform, or proximity spaces, the fundamental concept being indexed neighborhoods of sets. None of the assertions are proved. Most of them are simple, but the assertion on page 404 that a product of polytopological spaces is polytopological is false. (The product of two closed real intervals fails to satisfy Axiom Q.) *J. Isbell* (Seattle, Wash.)

1290:
Atsuji, Masahiko. A space in which every continuous real function is uniformly continuous (general case). Solution of problem 6.2.16. Sûgaku **8** (1956/57), 211–213. (Japanese)

The author gives a necessary and sufficient condition (rather complicated) for a uniform space to have the property that every continuous real function is uniformly continuous. *I. Amemiya* (Kingston, Ont.)

1291:
Jůza, Miloslav. Remark on complete metric spaces. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **6** (1956), 143–148. (Slovak. Russian summary)

The real line (with its usual uniformity) is metrized in such a way that there exists a decreasing sequence of closed spheres with a void intersection.

M. Katětov (Prague)

1292:
Corson, H. H. The determination of paracompactness by uniformities. Amer. J. Math. **80** (1958), 185–190.

All spaces considered are assumed to be Hausdorff. The author defines a filter $\mathscr{F}$ to be weakly Cauchy relative to a uniformity $\mathscr{U}$ if for every $U \in \mathscr{U}$, some filter containing $\mathscr{F}$ has a member small of order U. Theorem 1: The following are equivalent. (a) X is paracompact. (b) The set of all neighborhoods of the diagonal in $X \times X$ is a uniformity for X, and the product of X with every compact space is normal. (c) There is a uniformity for X under which every weakly Cauchy filter has a cluster point. (d) If $\mathscr{F}$ is a filter on X whose image has a cluster point in any metric space into which X is continuously mapped, then $\mathscr{F}$ has a cluster point in X. In particular, (b) implies (a); the converse is a well-known result of Dieudonné [J. Math. Pures Appl. (9) **23** (1944), 65–76; MR **7**, 134]. Theorem 2: The following are equivalent for regular X. (a) X is a Lindelöf space. (b) Every filter, weakly Cauchy in the smallest uniformity under which all continuous real-valued functions are uniformly continuous, has a cluster point. (c) If $\mathscr{F}$ is a filter whose image has a cluster point in the reals under every real continuous image of X, then $\mathscr{F}$ has a cluster point in X. Theorem 3: Let X be topologically complete (i.e., complete in the largest admissible uniformity $\mathscr{U}$). Then X is paracompact if and only if every filter weakly Cauchy relative to $\mathscr{U}$ is contained in a maximal weakly Cauchy filter. {Reviewer's remark: The author mentions J. L. Kelley's conjecture that a completely regular space is paracompact provided that it is topologically complete and the set of all neighborhoods of the diagonal is a uniformity. In 1956, M. J. Mansfield discovered an error in I. Namioka's proposed counterexample. Thus, Kelley's conjecture is apparently still open.} *L. Gillman* (Princeton, N.J.)

1293:
Kuratowski, K. Quelques propriétés de l'espace des ensembles LC^n. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 967–974, LXXXI. (Russian summary)

Let Y be a complete metric space; the family $LC^n(Y)$ of all its compact LC^n subsets can be made into a complete metric space [Kuratowski, Fund. Math. **43** (1956), 114–138; MR **18**, 58]. It is shown here that if $A \in LC^n(Y)$, X is a perfect compact space, $\dim X \leqq n+1$, and $f: X \to A$ is continuous, $f(X)=A$, then for every $\varepsilon>0$ there is a neighborhood $\mathfrak{A}$ of A in $LC^n(Y)$ such that for every $B \in \mathfrak{A}$ there is a $g: X \to B$ with $g(X)=B$ and $|g-f|<\varepsilon$. As consequence: If $A \in LC^n(Y)$, $\dim A=m \leqq n+1$, there is a neighborhood of A in $LC^n(Y)$ composed exclusively of sets with dimension $\geqq m$. More generally, if $A \in LC^n(Y)$ has a property invariant under "small transformations" [Kuratowski, Topologie, v. II, Warszawa, 1952; MR **14**, 892], there is a neighborhood of A in $LC^n(Y)$ composed exclusively of sets having the same property. *J. Dugundji* (Los Angeles, Calif.)

1294:
Griffiths, H. B. Borel sets and countable series of operations. Fund. Math. **44** (1957), 115–155.

The paper deals with a problem of Dienes [J. Math. Pures Appl. (9) **26** (1947), 227–235; MR **9**, 573]: given two Borel sets X, Y in a space E, is it "decidable D" which of the cases $X=Y$, $X \subseteq Y$, $X \supseteq Y$, $X \cap Y=0$, $X \cap Y \neq 0$ holds? In the simplest cases "decidability D" means the possibility of decision by means of comparison of size of integers. On the other hand the paper deals with complete separable metric spaces and presents a new approach to them inasmuch as the following problem is solved: "Given an abstract countable partially-ordered set, what axioms must the ordering satisfy in order that the set be order-isomorphic to a basis of open sets of some complete separable metric space?" Let $\mathfrak{B}=\{0, 1\}$, and $I=\{0, 1, 2, \cdots\}$; the main notions are the "$\mathfrak{B}$-function" and the corresponding "threads". Any mapping $A: I^4 \to \mathfrak{B}$ is called a $\mathfrak{B}$-function. If for $(n, i, j) \in I^3$ we have $A_n(i, j)=A(n, i, n+1, j)=1$, then (n, i, j) is called a segment of A. If for a mapping $\mathfrak{t}: I \to I$ we have $A_n(t_n, t_{n+1})=1$, then $\mathfrak{t}$ is called a thread in A and one writes $\mathfrak{t} \subseteq A$; here $t_n=\mathfrak{t}(n)$. For threads A, B the relation $A \subseteq B$ or $B \supseteq A$ is defined to mean that every segment of A is a segment of B. A is reduced if for every segment $(0, i, j)$ there exists a thread $\mathfrak{t}$ in A satisfying $t_0=i$, $t_1=j$; A is fully reduced if ${}^qA$ is reduced for every $q \in I$; ${}^qA$ is defined so that ${}^qA(l, m, n, p)=1$ means $A(l+q, m, n+q, p)=1$. Many further notions are introduced and examined. In particular, by a subthread $\mathfrak{t}'$ of $\mathfrak{t}$ one means a mapping $t': I \to I$ such that for every $n \in I$, $\mathfrak{t}'(n)=\mathfrak{t}(p(n))$, where $p: I \to I$ is monotonic increasing, and $p(n)>n$. In order that $\mathfrak{t}'$ be a thread in A, one restricts A by the conditions C_2-C_4. E. g., C_3 means the transivity: if $A(i, j, i+p, q)=1=A(i+p, q, i+p+n, m)$ $(p, n \geqq 0)$, then $A(i, j, i+p+n, m)=1$. One considers conditions C_1-C_9. For threads $\mathfrak{s}$, $\mathfrak{t}$ in A one defines $\mathfrak{s}<\mathfrak{t}$ or $\mathfrak{t}>\mathfrak{s}$, provided $A^n(t_n, s_n)=1$ $(n \in I)$; here $A^n(i, j)=A(n, i, n, j)$. If one writes $\mathfrak{s}=\mathfrak{t}$ if and only if there exists a thread $\mathfrak{u}$ in A satisfying $\mathfrak{u}<\mathfrak{s}$ and $\mathfrak{u}<\mathfrak{t}$, one gets an equivalence relation (Theorem 3.9), whose classes yield a topological space $\mathbf{A}=\mathbf{A}(A)$: one defines u_{ij} $(i, j \in I)$ as the system of all the classes $[\mathfrak{t}]$ such that for some thread $\mathfrak{s} \in [\mathfrak{t}]$ $\mathfrak{s}(i)=j$; otherwise $u_{ij}=0$; the sets u_{ij} serve as basis for G-sets ($[\mathfrak{t}]$ is the element of $\mathbf{A}$ in which $\mathfrak{t}$ lies). By imposing on A conditions C_1-C_9, the space $\mathbf{A}(A)$ is a separable complete metric space. Topologically, one gets in this way every complete separable metric space (Th.

4.16). In the language of $\mathfrak{B}$-functions the author gives a representation of Borel sets in such a space and proves that the problem of deciding whether or not a given B-set is empty is decidable D (the definition of decision D is too long to be reproduced here). Since the notion of A is fundamental, let us reproduce its definition for a complete separable metric space S. Let $b_0, b_1, \cdots$ be a dense set in S, and $W_0, W_1, \cdots$ an enumeration of the open basis $K(b_j,(1+n)^{-1})=\{x|x\in S;\ \rho(b_j,x)<(1+n)^{-1}\}$ $(j,n\in I)$. Define $A(n,p,m,q)=1$ if and only if $m\geqq n$, diam $W_p\leqq 2/(1+n)$, diam $W_q\leqq 2/(1+m)$, and either $p=q$ and $m\in\{n,n+1\}$, or $p\neq q$ and $W_p\supset\overline{W}_q$; put $A(n,p,m,q)=0$ otherwise. The corresponding space $\mathbf{A}(A)$ is isomorphic to S. *Đ. Kurepa* (Zagreb)

1295:

Rodniansky, A. M. The mapping of a topological space multiplied by Euclidean space into Euclidean space. Dokl. Akad. Nauk SSSR **115** (1957), 659–662. (Russian)

The principal results are sufficient conditions for local openness and invertibility of k-fold $(k\geqq 0)$ differentiable mappings $f_x(y)$, y and $f_x(y)$ in a Euclidean space, x in a topological space. They extend previous work of the author [Mat. Sb. N.S. **36(78)** (1955), 233–262; MR **16**, 912] and of Kudryavcev [Uspehi Mat. Nauk (N.S.) **9** (1954), no. 3(61), 155–156; MR **16**, 121].

J. Isbell (Seattle, Wash.)

1296:

Deleanu, Aristide. Sur un théorème de point fixe. Com. Acad. R. P. Romîne **7** (1957), 839–844. (Romanian. Russian and French summaries)

Let E and T be sets, and let $F=R^T$ with its usual partial order (R=reals). The author terms "general metric on E" a mapping $\delta: E\times E\to F$ satisfying the usual conditions imposed on a numerically-valued metric (T=a one-point set). The fixed-point theorem referred to states that if $f: E\to E$ satisfies a Lipschitz condition

$$\delta(f(x),f(y))\leqq\mu\cdot\delta(x,y)$$

for some $\mu\in F$ satisfying $0\leqq\mu(t)<1 (t\in T)$, then f admits a unique fixed point $x_0\in E$. The usual proof for metric spaces is adapted with only verbal changes. Corollary 1: If $f: F\to F$ and $|f\xi(t)-f\eta(t)|\leqq\mu(t)\cdot|\xi(t)-\eta(t)|$ $(\xi,\eta\in F;\ t\in T;\ \mu\in F$ as above), then f admits a unique fixed point $\xi_0\in F$. (Take $\delta(\xi,\eta)=|\xi-\eta|$.) From this the author derives existence and uniqueness theorems for differential and integral equations involving functions with values in a complete, separated, locally convex space S which extend those known for Banach spaces (and which depend on the fixed-point theorem for metric spaces). The use of general metric spaces makes this extension somewhat neater than it would otherwise be.

R. E. Edwards (Woking)

1297:

Bourgin, D. G. Fixed points on neighborhood retracts. Rev. Math. Pures Appl. **2** (1957), 371–374.

The author obtains a number of fixed point theorems for "sufficiently regular spaces which have had a finite number of holes punched out". The main theorem is the following. Let X be an AR and let $Y_1, \cdots, Y_n$ $(n>1)$ be open subsets whose closures are disjoint AR's, and let G be their union. Then every map $(X-G)\to X$, such that $Y_i\to\overline{Y}_i$ for each i, has a fixed point.

P. A. Smith (New York, N.Y.)

1298:

Nagata, Jun-iti. A generalization of a theorem of W. Hurewicz. J. Inst. Polytech. Osaka City Univ. Ser. A. **9** (1958), 37–38.

Let "ind dim" denote inductive dimension defined in terms of boundaries of neighborhoods of closed sets and "dim" denote covering dimension. The author proves the following generalization of a theorem of Hurewicz: If f is a closed mapping of a normal space R onto a perfectly normal space S such that for each point q of S the boundary of $f^{-1}(q)$ contains at most $m+1$ points $(m\geqq 0)$, then ind dim $S\leqq$ ind dim $R+m$. K. Morita [Proc. Japan Acad. **32** (1956), 161–165; MR **18**, 141] has proved the weaker inequality dim $S\leqq$ ind dim $R+m$ assuming that $f^{-1}(q)$ contains at most $m+1$ points and that S is normal (instead of perfectly normal). *E. Dyer* (Chicago, Ill.)

1299:

Whyburn, G. T. Dimension and non-density preservation of mappings. Pacific J. Math. **7** (1957), 1243–1249.

This paper is concerned primarily with providing proofs of theorems announced in Sections 13 and 14 of an expository address of the author [Bull. Amer. Math. Soc. **62** (1956), 204–218; MR **17**, 1229].

E. Dyer (Chicago, Ill.)

1300:

Whyburn, G. T. Quasi-open mappings. Rev. Math. Pures App. **2** (1957), 47–52.

The author shows that a mapping of a region in the plane into the plane is quasi-open if and only if the mapping possesses the minimum modulus property. The following theorem is then proved. Let X and Y be locally connected generalized continua and let the sequence of quasi-open mappings $f_n: X\to Y$ converge almost uniformly to the mapping $f: X\to Y$. For any $y\in Y$ and any open set V in X containing a non-empty compact component K of $f^{-1}(y)$ there exists a conditionally compact open set U in X with $K\subset\overline{U}\subset V$, a region R in Y about y and an integer N such that, for each $n>N$, U contains a component Q_n of $f_n^{-1}(R)$ with $f_n(Q_n)=R$. It is then shown that many important theorems are nearly immediate consequences; e.g.: the limit mapping $f: X\to Y$ is quasi-open; if each f_n is weakly monotone and $f^{-1}(y)$ has a non-empty compact component for each $y\in Y$, then f is compact and monotone; if f is light, the sequence is uniformly approximately open on every compact set A in X. Finally the relationship of the theorem to one of J. W. T. Youngs (any monotone mapping of a compact 2-manifold onto itself can be approximated uniformly closely by homeomorphisms) [Duke Math. J. **15** (1948), 87–94; MR **9**, 524] is discussed.

C. J. Titus (Berkeley, Calif.)

ALGEBRAIC TOPOLOGY

See also 868, 1339, 1340.

1301:

Aleksandrov, A. D. On completion of a space by polyhedra. Vestnik Leningrad. Univ. Ser. Mat. Fiz. Him. **9** (1954), no. 2, 33–43. (Russian)

1302:

Venkov, B. A. On a class of Euclidean polyhedra. Vestnik Leningrad. Univ. Ser. Mat. Fiz. Him. **9** (1954), no. 2, 11–31. (Russian)

1303:

Ryshkov, S. S. On the combinatorial topology of Hilbert space. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 494–497. (Russian)

Continuing the work of Boltyanskiĭ on the homology theory of Hilbert space [same Dokl. (N.S.) **105** (1955), 1141–1143; MR **17**, 883], the author proves the following invariance theorem. If K is a cellular decomposition of Hilbert space H and K_1 is a closed subcomplex of K, denote by $\|K_1\|$ the closure of the (point set) union of the cells of K_1. Let K, K' be any two cellular decompositions of H, and let P be a subset of H such that $P=\|K_1\|=\|K_2\|$, where K_1, K_2 are closed subcomplexes of K, K', respectively. Then ${}_rH(K_1)\approx{}_rH(K_2)$ (${}_rH(M)$ is the homology group of M of defect r over an arbitrary coefficient group) for every r. To prove the theorem, the author introduces the notions of a cellular mapping relative to a pair of cells of defect r, and the degree of a mapping, defined for a certain class of admissible cellular mappings. If K_1 and K_2 are closed subcomplexes of K, K', with $\|K_1\|\subset\|K_2\|$, and i: $\|K_1\|\to\|K_2\|$ is the inclusion map, it is possible to define (by induction on the defect r) a sequence of cellular approximations i_r: $\|K_1\|\to\|K_2\|$ to i which turn out to be admissible cellular mappings relative to an arbitrary pair of cells of defect r of K_1, K_2, respectively. The induced homomorphism ${}_*i$: ${}_rL(K_1)\to{}_rL(K_2)$ (the group of chains of defect r) can then be defined and the isomorphism of the homology groups proved in the usual way.

A consequence of the invariance theorem is that the exterior homology group of defect r of a closed subset of H, defined as the limit of the inverse system of homology groups of defect r of a monotonic sequence of closed subcomplexes containing the given set and closing down on it, is an invariant.

The author remarks that analogous results can be obtained for the cohomology groups.

H. Komm (Troy, N.Y.)

1304:

Ryshkov, S. S. On a class of continuous mappings of some ∞-dimensional sets. Dokl. Akad. Nauk SSSR (N.S.), **114** (1957), 961–963. (Russian)

(See the review of the same author's paper above.) In this paper, a class of mappings of subsets of Hilbert space, called a Λ-class, is defined. The purpose is to show that the H-polyhedra (the point set closures of infinite-dimensional complexes in Hilbert space H) homeomorphic in this class have isomorphic infinite-dimensional homology groups. In particular, the class contains every mapping φ: $M\to H$ ($M\subset H$) of the form $\lambda e+a$, where λ is any real number, a is a continuous mapping of M into a compact subset of H, and e is the identity. Two invariance theorems are stated. The first is the one mentioned above. The second formulates the invariance, in the class Λ, of the exterior homology groups of closed subsets of H defined in the author's previous paper. The proofs of these theorems are not given, but the author states two lemmas (also without proof) which he remarks are central to the proof of the theorems. *H. Komm* (Troy, N.Y.)

1305:

Frum-Ketkov, R. L. The behaviour of cycles in the continuous mapping of compacts. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 249–252. (Russian)

Let M_n and M_n' be closed orientable n-dimensional manifolds and let f be a continuous mapping of M_n into M_n' which carries the fundamental n-cycle of M_n to the zero cycle in M_n'. The author shows that if the coefficient group is the group of rational numbers, then for each k, $0\leq k<n-1$, there is in M_n a k-dimensional essential cycle carried by f to the zero cycle in M_n'. Here, an essential cycle is one which is not homologous to zero in at least one of its carriers. The proof works down one dimension at a time by induction; the compact sets of the title are the carriers of the cycles.

M. M. Day (Urbana, Ill.)

1306:

***Whitehead, George W. The homology suspension.** Colloque de topologie algébrique, Louvain, 1956, pp. 89–95. Georges Thone, Liège; Masson & Cie, Paris, 1957. 375 fr. belges; 3000 fr. français.

The first section of the paper is essentially a review of the author's earlier results [Ann of Math. (2) **62** (1955), 254–268; MR **17**, 520]. In the second section a construction is given which to each n-connected space X assigns a suitably filtered space W. This construction is, in some not very clearly specified way, analogous to the bar construction. In the third section the Lusternik-Schnirelman category of X is related to some properties of W.

S. Eilenberg (New York, N.Y.)

1307:

***Papy, Georges. Relations entre les complexes de de Rham et d'Alexander.** Colloque de topologie algébrique, Louvain, 1956, pp. 151–165. Georges Thone, Liège; Masson & Cie, Paris, 1957. 375 fr. belges; 3000 fr. français.

A general discussion of various graded differential algebras and their relation to the de Rham theorem and the Leray uniqueness theorem.

W. Ambrose (Cambridge, Mass.)

1308:

Teleman, Silviu. Sur la formule d'Euler-Poincaré-Hopf. Rev. Math. Pures Appl. **2** (1957), 551–554.

Given a linear transformation t: $V\to V$ of a finite-dimensional vector space V over the field F, let $P[t, V]$ denote the characteristic polynomial of t (i.e. $\det(a_{ij}-x\delta_{ij})$, where (a_{ij}) is a matrix of t). If W is a subspace of V, $tW\subset W$, let $t|W=t_0$, $t^*:V/W\to V/W$ be induced by t. Then $P[t, V]=P[t^*, V/W]\cdot P[t_0, W]$. Now let $f:K\to K$ be a simplicial mapping of a finite complex K in itself, which induces endomorphisms f_r, f_{*r} of the chain-groups $C_r(K, F)$, $H_r(K, F)$ respectively. The above equation then implies the formula

$$\prod_{r=0}^{n}\{P[f_r, C_r(K, F)]\}^{\theta(r)}=\prod_{r=0}^{n}\{P[f_{*r}, H_r(K, F)]\}^{\theta(r)},$$

where $\theta(r)=(-1)^r$. An immediate deduction is the formula of Euler-Poincaré-Hopf. *H. B. Griffiths* (Bristol)

1309:

Zeeman, E. C. The lack of universal coefficient theorems for spectral sequences. Proc. Cambridge Philos. Soc. **53** (1957), 925–927.

The author shows, by an example, that the spectral sequence of a double complex with a coefficient group is not determined by the spectral sequence with integral coefficients as the coefficient group.

V. Gugenheim (Baltimore, Md.)

1310:

Yamanoshita, Tsuneyo. On the homotopy groups of spheres. Jap. J. Math. **27** (1957), 1–53.

The author does the complete and detailed computations to prove the following result. The 2-primary components of $\pi_i(S^3)$ are: Z_2 for $i=5$, Z_4 for $i=6$, Z_2 for $i=7$, Z_2 for $i=8$, 0 for $i=9$ and 10, Z_2 for $i=11$, Z_2+Z_2

for $i=12$, Z_4+Z_2 for $i=13$, $Z_4+Z_2+Z_2$ for $i=14$, Z_2+Z_2 for $i=15$, Z_2 for $i=16$, Z_2 for $i=17$, Z_2 for $i=18$, Z_2+Z_2 for $i=19$, $Z_4+Z_2+Z_2$ for $i=20$, and $Z_4+Z_2+Z_2$ for $i=21$. The method is that of killing homotopy groups. (The reader will be amazed at the length of some of the lemmas and propositions. For example, Lemma 2.52 contains 516 symbols of the form Sq^i.)

F. P. Peterson (Cambridge, Mass.)

1311:

Postnikov, M. M. Classification of continuous mappings of an n-dimensional polyhedron into a connected topological space, aspherical in dimensions greater than unity and smaller than n. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **22** (1956), 165–202. (Russian)

This paper is concerned with cohomology theory with local coefficients, and extension and classification theorems for mappings.

The first chapter deals with cohomology theory with local coefficients [N. E. Steenrod, Ann. of Math. (2) **44** (1943), 610–627; MR **5**, 104], cohomology theory in noncommutative groups [H. Robbins, Trans. Amer. Math. Soc. **49** (1941), 308–324; MR **3**, 141], and cohomology theory in abstract groups [S. Eilenberg and S. MacLane, Ann. of Math. **48** (1947), 51–78; MR **8**, 367]. Cohomology theory over cell-complexes which are sub-complexes of $K\times I$, where K is a simplicial locally finite complex and I is the unit interval, is presented in a somewhat different form from that of the original papers, suitable for applications in Chapters II and III.

Chapter II contains basic definitions of homotopy theory as well as the definition of obstruction. (The obstruction to the extension of a mapping f, of $L^p \subset L$ into a space Y, to the subcomplex L^{p+1} is a cochain of L^{p+1} with coefficients in the p-dim. homotopy group π^p of Y: $c_f{}^{p+1}(\sigma^{p+1})$ is the element of π^p defined by the image of the boundary of σ^{p+1} under the mapping f.) For two mappings f_0 and f_1, defined on K^p and coinciding on K^{p-1}, one defines the mapping (f_0, f_1) on $A=K^p\times 0\cup K^p\times 1\cup K^{p-1}\times I$ by $(f_0, f_1)(x, i)=f_i(x)$ $(i=0, 1)$ for $x\in K^p$, and $(f_0, f_1)(x, t)=f_0(x)=f_1(x)$ $(t\in I)$ for $x\in K^{p-1}$. Then for every simplex σ^p of K^p one writes $\theta c^p(\sigma^p\times I)=c^p(\sigma^p)$ for every cochain c^p. Let $c_{(f_0,f_1)}{}^{p+1}$ be the obstruction determined by two continuous mappings defined on K^p, coinciding on K^{p-1} and mapping all vertices of K^p into one point. Then there exists a cochain d^p of K^p with coefficients in π^p such that $\theta d^p=c_{(f_0,f_1)}{}^{p+1}$. The cochain d^p is called the cochain distinguishing f_0 and f_1. The author shows that f_0 is homotopic to f_1 rel. K^{p-1} if and only if $d^p=1$.

The principal results of the paper are contained in Chapter III and consist in necessary and sufficient conditions that (a) a normal mapping (i.e. one which maps all vertices into one point) of $A\overline{C}K$ into Y be extendable to K, where $\dim K=n+1$ and Y is as in the title; and (b) two normal mappings of an n-complex into Y, agreeing on a subcomplex A, be homotopic rel. A.

An appendix to the paper derives some results of I. I. Gordon [Dokl. Akad. Nauk SSSR (N.S.) **65** (1949), 441–444; MR **10**, 617] from the theorems proved in the paper. The paper has at least 13 misprints in subscripts and other notation.

C. Masaitis (Aberdeen, Md.)

1312:

***Postnikov, M. M. Investigations in the homotopy theory of continuous mappings. I. The algebraic theory of systems. II. The natural system and homotopy type.** Amer. Math. Soc. Transl. (2) **7** (1957), 1–134.

The original, Trudy Mat. Inst. Steklov., no. 46, Moscow 1955, was reviewed in MR **17**, 652.

1313:

Boltyanskiĭ, V. G. Homotopy theory of continuous mappings and of vector fields. Amer. Math. Soc. Transl. (2) **7** (1957), 135–321.

The original, Trudy Mat. Inst. Steklov., no. 47, Moscow, 1955, was reviewed in MR **17**, 1232.

1314:

Weier, J. Über dimensionale Verschiebung. Fund. Math. **45** (1958), 130–137.

The author is concerned with the following question. Let P and Q denote connected closed manifolds, $a\in Q$, and F a homotopy class of mappings of P into Q. Is it true that for $f\in F$ and $\varepsilon>0$, there exists a $g\in F$ such that (1) $g^{-1}(a)$ is a polyhedron, (2) $\dim g^{-1}(a)=\dim f^{-1}(a)$, and (3) there is an ε-mapping of $f^{-1}(a)$ into $g^{-1}(a)$? He settles this affirmatively for the cases $\dim P=n$, $Q=S_n$ or S_{n-1} and $\dim f^{-1}(a)=0$. The arguments construct the homotopies using the local affine structure of the manifolds.

E. Dyer (Chicago, Ill.)

1315:

Ganea, Tudor. Relations between category and n-dimensional category. Rev. Math. Pures Appl. **2** (1957), 327–329.

Let X be an arcwise connected semi-locally contractible space. Call an open subset A of X an n-categorical set if any map of a finite n-dimensional complex into A is contractible in X, and define $\operatorname{cat}_n X$ to be the least integer $k\leq\infty$ such that X admits a cover consisting of $k+1$ open n-categorical subsets. Obviously $\operatorname{cat}_1 X\leq \operatorname{cat}_2 X\leq\cdots$ ($\leq \operatorname{cat} X$, where $\operatorname{cat} X$ denotes 1 less than the Lusternik-Schnirelmann category of X). Th. 1: If $\pi_{n+1}(X)$ is trivial, then $\operatorname{cat}_n X=\operatorname{cat}_{n+1} X$; hence, if X is aspherical, $\operatorname{cat}_1 X=\operatorname{cat}_2 X=\cdots$. Th. 2: If X is paracompact and dominated by some CW-complex and if X is aspherical then the minimum of the dimensions of the CW-complexes that dominate X is equal to $\operatorname{cat}_1 X=\operatorname{cat} X$.

R. H. Fox (Princeton, N.J.)

1316:

Schlesinger, James W. On transitive translation functions. Proc. Amer. Math. Soc. **9** (1958), 507–510.

Let p be a continuous map of the space E onto B. According to the definition of W. Hurewicz [Proc. Nat. Acad. Sci. **41** (1955), 956–961; MR **17**, 519] the triple (E, B, p) is a fibre space if there exists a continuous lifting function λ which assigns to each pair (e, ω) (consisting of a point $e\in E$ and a path ω in B starting at the point $p(e)$) a path $\lambda(e, \omega)$ in E starting at e such that $\lambda(e, \omega)$ projects onto ω. For any two points $b_0, b_1\in B$ and any path ω from b_0 to b_1, one may define a "translation function" $\tau(\omega): p^{-1}(b_0)\to p^{-1}(b_1)$ by the formula $[\tau(\omega)](e)=[\lambda(e, \omega)](1)$ for any $e\in p^{-1}(b_0)$. This translation function is said to be transitive if for any three points $b_0, b_1, b_2\in B$ and paths ω_1 from b_0 to b_1 and ω_2 from b_1 to b_2, $\tau(\omega_1\omega_2)=\tau(\omega_2)\circ\tau(\omega_1)$ (where $\omega_1\omega_2$ is the path obtained by traversing ω_1 and ω_2 in succession).

This paper is concerned with the question of when it is possible to choose the lifting function λ for a given fibre space (E, B, p) so that the associated translation function is transitive. It follows easily from known results that such a transitive translation function always exists for fibre bundles with a totally disconnected structural group [see N. E. Steenrod, The topology of fibre bundles, Princeton University Press, 1951; MR **12**, 522; especially

§ 13]. The main result of the present paper is a sort of converse of this known fact, viz.; if a fibre bundle over a finite polyhedron has structural group G with no small subgroups and has a transitive translation function, then it is equivalent in G to an H-bundle, where H is a totally disconnected subgroup of G.

W. S. Massey (Providence, R.I.)

1317:

Adachi, Masahisa; and Shimada, Nobuo. On tangent structures of lower dimensional manifolds. Proc. Japan Acad. **34** (1958), 417.

The authors consider the problem of whether or not two different differentiable structures on a compact orientable m-manifold M^m can have equivalent tangent bundles. They state the following two theorems without proof. Th. 1: Let $m=5$, and suppose $H^4(M)$ has no 2-torsion and that $W^4\neq 0$. Then its tangent bundles are equivalent if and only if the corresponding 4-dimensional Pontrjagin classes are equal. Th. 2: Let $6\leqq m\leqq 8$, and suppose $H^4(M)$ has no 2-torsion. Then its tangent bundles are equivalent if and only if the corresponding 4-dimensional Pontrjagin classes are equal. *F. P. Peterson* (Cambridge, Mass.)

1318:

Rohlin, V. A. On Pontrjagin characteristic classes. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 276–279. (Russian)

The index of a differentiable orientable compact manifold M_{4k} can be expressed as $\alpha_k^{-1}\cdot\varphi_k(P_1,\cdots,P_k)$, where the P_i are the Pontrjagin classes, φ_k is a certain polynomial with integral, relatively prime, coefficients, and α_k is a certain integer (Hirzebruch). Denoting by p_k the weak Pontrjagin classes, i.e., modulo torsion, put $s_k=s_k(M_{4k})=\varphi_k(p_1,\cdots,p_k)$; this is then a topological invariant. The main result of the present note is that the class $s_k(M_n)$, which is defined for any n, is topologically invariant also for $n=4k+1$. For the proof, suppose M_n has two differentiable structures. Let u_{4k} be any integral $4k$-cycle, and let V resp. V_1 be differentiable submanifolds of M_n representing u_{4k} in the two structures. The theorem follows easily if the indices of V and V_1 can be shown to be equal. It is stated that V_1 can be chosen so as to lie in a small neighborhood of V and not to intersect V. But then V and V_1 form the boundary of an n-dimensional submanifold of M_n, and therefore their indices are equal.

It is noted that $s_k(M_{4k+1})$ is not determined abstractly by the cohomology ring, with an example of a 5-manifold constructed by Wu. Further $s_k(M_{4k+1})$ is divisible by α_k. Finally, if the second Stiefel class of M_5 is 0, then $p_1(M_5)$ is divisible by 48. *H. Samelson* (Ann Arbor, Mich.)

1319:

Smale, Stephen. Regular curves on Riemannian manifolds. Trans. Amer. Math. Soc. **87** (1958), 492–512.

This paper proves a major generalization of the following theorem of Whitney: Two regular closed curves on the plane are freely regularly homotopic if and only if they have the same rotation number [Compositio Math. **4** (1937), 276–284]. The generalization proved here is: Let x_0 be a point of the unit tangent bundle T of a Riemannian manifold M. Then there is a 1-1 correspondence between the set π_0 of classes (under regular homotopy) of regular curves on M which start and end at the point and direction determined by x_0, and $\pi_1(T, x_0)$.

A fundamental theorem used in the proof of the above is: Let E be the space of all regular curves on M starting at the point and direction determined by $x_0\in T$. Define $\pi: E\to T$ by sending a curve into the tangent of its endpoint at its endpoint. Then the triple (E, π, T) has the covering homotopy property for polyhedra.

If ω is a 1-form on T, it has an integral curve. Every curve in T which is lifted over M is such an integral curve, but not conversely. The following theorem is proved: Let ω be a 1-form of Class A on M^3 such that $\omega\wedge d\omega\neq 0$ on M^3, and let $x_0\in M^3$. Denote by Ω_ω the loop space at x_0 of piecewise regular curves on M which are integral curves of ω, and by Ω the ordinary loop space of M^3 at x_0. Then the inclusion $i: \Omega_\omega\to\Omega$ is a weak homotopy equivalence. *C. B. Allendoerfer* (Seattle, Wash.)

1320:

Lefschetz, Solomon. On coincidences of transformations. Bol. Soc. Mat. Mexicana (2) **2** (1957), 16-25.

The author's well-known theorem on coincidences of mappings of one ANR into another is here refined and re-proved. The refinement lies in that the mappings considered are allowed to be upper semi-continuous set-valued and also in that the single number (the alternating sum of the traces of certain matrices) whose non-vanishing ensures the existence of a coincidence is here replaced by a matrix, the non-vanishing of any of whose elements guarantees a coincidence. The proof here is modeled on the author's original proof, but is less algebraic. It is carried out first for finite geometric complexes and then extended to ANR's by imbedding them in a Hilbert parallelotope. *E. G. Begle* (New Haven, Conn.)

1321:

Kotzig, Anton. Euler lines and decompositions of a regular graph of even order into two factors of equal orders. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **6** (1956), 133–136. (Slovak. Russian summary)

Let G be a connected Euler graph with an even number of edges. Every Euler line of G generates, in an obvious way, a decomposition of the set of all edges of G into two sets H_1, H_2 such that every vertex is incident with the same number of edges from H_1 and from H_2. It is proved that every such decomposition is generated by some Euler line. The corresponding theorem on decompositions mentioned in the title of the note is obtained as a corollary.

M. Katětov (Prague)

1322:

Sabidussi, Gert. Graphs with given group and given graph-theoretical properties. Canad. J. Math. **9** (1957), 515–525.

The author establishes interesting generalizations of the theorem of R. Frucht [Compositio Math. **6** (1938), 239–250] that given any finite group G there exist infinitely many non-isomorphic connected graphs whose automorphism group is isomorphic to G.

It is shown that, given any finite group G, for each of the four properties (1)–(4) stated below, there exist infinitely many non-homeomorphic finite connected fixed-point-free graphs without loops and multiple edges, whose automorphism group is isomorphic to G, and which in addition possess the property. "Fixed-point-free" means that there is no vertex of the graph which is invariant under all automorphisms of the graph.

The four properties are: (1) The connectivity of the graphs is n, where n is any given integer $\geqq 1$; (2) the chromatic number of the graphs is n, where n is any given integer $\geqq 2$; (3) the graphs are regular of degree n, where n is any given integer $\geqq 3$; (4) the graphs are spanned by a graph which is homeomorphic to a given connected graph.

Constructions which furnish graphs of the required kind are given. They use a method of deriving a graph from two given graphs called graph multiplication by the author. The graphs constructed are not the only ones having the required characteristics; it was not the aim of the author to discover all such graphs but rather to demonstrate the usefulness of his graph multiplication. Use is made of a theorem on graph multiplication in a paper by the author which will appear in Math. Z.

G. A. Dirac (Hamburg)

1323:

Berge, Claude. Two theorems in graph theory. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 842–844.

A set X of vertices of a graph G is called internally stable if no two are joined by an edge. It is called a cover of G if each edge is incident with some member of X. A set V of edges is called a matching if no vertex is incident with two distinct members of V. The author considers the problems of finding an internally stable set with the maximum number of members, a cover with the minimum number, and a matching with the maximum number. The author's two theorems give algorithms for the solution of all three problems. The first also gives a test for determining whether or not a given matching is maximum.

W. T. Tutte (Toronto, Ont.)

DIFFERENTIAL GEOMETRY, MANIFOLDS

See also 970, 993, 994, 1080, 1295, 1318, 1604.

1324:

Melzi, Giovanni. Su alcune trasformazioni puntuali tra spazi ordinari estendenti le trasformazioni conformi. Rend. Mat. e Appl. (5) **16** (1957), 96–117.

It is well known that every biregular correspondence T, of class C^1, between two (regions of two) euclidean spaces E_n, E_n', of dimension n, associates at each point P of E_n an hyperellipsoid of deformation, $\mathbf{I}_P$, with centre P [cf., e.g., B. Segre, Some properties of differentiable varieties and transformations, Springer, Berlin, 1957; MR **19**, 679; § 1]. The present paper deals with the case when $n=3$ and each $\mathbf{I}_P$ is a quadric of revolution.

It is shown that the problem of finding all the correspondences T satisfying these conditions is equivalent to the one of obtaining the three-dimensional varieties of an E_6 whose tangent spaces are bitangent with respect to a certain V_3^4 belonging to the hyperplane at infinity of E_6; and the Pfaff equations determining those varieties are written. Special attention is paid to the case when the congruence formed by the curves touching, at each point P, the axis of revolution of $\mathbf{I}_P$ is orthogonal; the particular correspondences T thus defined are then investigated: they depend on six arbitrary functions of two variables, and may be considered as an extension of the conformal transformations.

B. Segre (Rome)

1325:

Vincensini, Paul. Vue d'ensemble sur l'oeuvre géométrique de Luigi Bianchi. Univ. e Politec. Torino. Rend. Sem. Mat. **16** (1956/57), 115–157.

Where other authors [e.g. G. Fubini, Ann. Mat. (4) **6** (1929), 45–83] have given a general survey of Bianchi's work, the present author confines himself to the more specific geometrical aspects, going into considerable detail. This paper thus becomes an introduction into many domains of classical differential geometry, notably into such subjects as the transformations of Bäcklund, the deformation of quadrics, stratifiable systems, W-congruences and B-surfaces, congruences of spheres and the rolling of surfaces upon others. The paper ends with a sketch of Bianchi's contributions to curved manifolds of more than three dimensions.

D. J. Struik (Cambridge, Mass.)

1326:

Dou S. I., Alberto. The symmetric representation of hexagonal four-webs. Collect. Math. **9** (1957), 41–58. (Spanish)

A four-web is called hexagonal if each of its four sub-three-webs is hexagonal. The hexagonal four-webs T_4 were studied by Mayrhofer [Math. Z. **28** (1928), 728–752; cf. Blaschke and Bol, Geometrie der Gewebe, Springer, Berlin, 1938, pp. 74–94]. In the present paper the author derives Mayrhofer's classification of the T_4's from his own symmetric representation of four-webs [Mem. Real Acad. Ci. Art. Barcelona **31** (1953), 133–218; MR **16**, 400]. The T_4's are characterized by the vanishing of the curvatures of their sub-three-webs. Thus a certain function $r=r(u, v)$ of the author's symmetric representation will satisfy certain partial differential equations (in particular, it will be harmonic). They readily imply that $\tan r(u, v)=U(u)/V(v)$ where $U''+2AU^3=V''+2AV^3=0$, $U'^2+AU^4=V'^2+AV^4=B$ (A, B=const). These equations yield Mayrhofer's classification: (i) U=const, V=const. In the (u, v)-plane T_4 consists of four pencils of parallel straight lines. All the Desargues configurations of T_4 are closed. (ii) $U'=V'$=const$\neq 0$. In the (U, V)-plane T_4 consists of the four pencils U/V=const, $U^2=V^2$=const, UV=const, U^2-V^2=const. The Desargues configurations corresponding to the first pencil are closed. (iii) $U=$sn u/dn u, $V=$sn v/dn v. The four pencils of T_4 can also be represented through Jacobi's elliptic functions. No Desargues configuration is closed.

P. Scherk (Saskatoon, Sask.)

1327:

Godeaux, Lucien. Sur les suites de Laplace inscrites dans un polièdre de Laplace. Bul. Inst. Politehn. Iaşi (N.S.) **3** (1957), no. 1–2, 7–10. (Russian and Romanian summaries)

1328:

Su, Buchin. The geometry of spaces with areal metrics. Math. Nachr. **16** (1957), 281–287.

A space with areal metric is a differentiable manifold in which the "area" of an m-dimensional submanifold is given a priori by an m-ple fundamental integral: $V=\int_R F(x^i, p_\alpha^i)du^1\cdots du^m$ ($i, j, k=1, 2, \cdots, n$; $\alpha, \beta, \gamma=1, 2, \cdots, m$), where $p_\alpha^i=\partial x^i/\partial u^\alpha$ and $x^i=x^i(u^\alpha)$ is a parametric expression of the submanifold in a coordinate neighborhood. Such a space is called an "areal space" following A. Kawaguchi [Tensor N.S. **1** (1950), 14–45; MR **12**, 536] and has been studied by many authors; see the references of the paper. The author obtains explicit forms of the first and second variations of the fundamental integral by making use of an affine connection satisfying certain axioms. It is shown, at the end, that the above result is a generalization of Davies' formula for Riemannian space.

A. Kawaguchi (Sapporo)

1329:

Fyadzenka, A. S. A method of passage to the limit in the theory of Riemann spaces. Vesci Akad. Navuk BSSR. Ser. Fiz.-Tèhn. Navuk **1958**, no. 2, 17–25. (Byelorussian)

1330:
Weiszmann, Andrei. Sur l'anisotropie des espaces généralisés. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. Fiz. **7** (1956), no. 1–4, 99–112. (Romanian. Russian and French summaries)

Author's summary: "Le présent mémoire démontre le caractère anisotrope des espaces continus à connexion généralisée. L'auteur résume les résultats de quelques travaux antérieurs en indiquant la méthode de la "variation de la métrique", étend cette méthode aux espaces généralisés par l'introduction d'une généralisation de la dérivation covariante et ainsi il démontre l'anisotropie de ces espaces. Les résultats s'appliquent à des cas spéciaux." *V. Hlavatý* (Bloomington, Ind.)

1331a:
Aragnol, André. Espace des tenseurs de type adjoint. Homomorphismes. C. R. Acad. Sci. Paris **244** (1957), 3014–3016.

1331b:
Aragnol, André. Espace des tenseurs de type adjoint. Existence des homomorphismes. C. R. Acad. Sci. Paris **244** (1957), 3124–3126.

1331c:
Aragnol, André. Espace des tenseurs de type adjoint. Théorèmes d'existence. C. R. Acad. Sci. Paris **245** (1957), 134–136.

1331d:
Aragnol, André. Notion d'espace fibré principal maximal. C. R. Acad. Sci. Paris **246** (1958), 3570–3572.

The author continues his investigations [same C. R. **244** (1957), 437–440, 2683–2686; MR **19**, 576] of connections in fibre bundles and of holonomy groups. The exposition is becoming more technical and compact. The reviewer is looking forward to a detailed coherent exposition, which will present in more digestible fashion the many interesting-looking features of these papers.

{Such a more elaborate treatment is given in the author's thesis [Université de Paris, 1958], just appeared in mimeographed form.} *A. Nijenhuis* (Seattle, Wash.)

1332:
Angelescu, Tr. Sur l'intégration d'une équation aux dérivées partielles. Com. Acad. R. P. Romîne **6** (1956), 1289–1292. (Romanian. Russian and French summaries)

By means of the substitution $1+x+y=t$, $\lambda(x, y)=\varphi(t)$, the equation

$$\frac{\partial^2\lambda}{\partial x\partial y}-\frac{1}{1+x+y}\left(e^{\lambda}\frac{\partial\lambda}{\partial x}+e^{-\lambda}\frac{\partial\lambda}{\partial y}\right)+\frac{(e^{\lambda}-e^{-\lambda})}{(1+x+y)^2}=0$$

is reduced to a Ricatti equation, which is solved for $\varphi(t)$, The equation appears in G. Vrănceanu, Com. Acad. R. P. Romîne **4** (1954), 335–340 [MR **17**, 406].
D. J. Struik (Cambridge, Mass.)

1333:
Prvanović, Mileva. Some properties of a family of conformal geodesics. Univ. Beogradu. Godišnjak Filozof. Fak. Novom Sadu **1** (1956), 313–318. (Serbo-Croatian. English summary)

1334:
Mishra, R. S. Sous-espaces non holonomes. Bull. Sci. Math. (2) **81** (1957), 17–21.

The author discuss the theory of non-holonomic subspaces in connection with the work by T. Takasu [Yokohama Math. J. **1** (1953), 1–28, 29–38, 39–74, 75–77, 79–82, 83–87; MR **15**, 350]. Let L_m be a differentiable manifold of dimension m and L_n an n-dimensional sub-manifold of L_m $(n<m)$. The author considers, in L_m, a non-holonomic system which defines a non-holonomic space with respect to L_m, and also, in L_n, another non-holonomic system which defines a non-holonomic space in L_n. The latter non-holonomic space is called a non-holonomic subspace of the former. The relations among the properties of both non-holonomic spaces are established. Finally the following result appears in the present note: If a curve C lies in L_n and the field of vectors is non-holonomic parallel along C with respect to L_m, it is also non-holonomic parallel with respect to L_n.
A. Kawaguchi (Sapporo)

1335:
Su, Buchin. Axiom of the plane in a descriptive geometry of K-spreads. Math. Nachr. **16** (1957), 215–226.

Let the differential equations of K-spreads be

$$\frac{\partial^2x^i}{\partial u^\alpha\partial u^\beta}+\Pi_{jk}{}^i(x, p)p_\alpha{}^jp_\beta{}^k=p_\gamma{}^jF_{\alpha\beta}{}^\gamma(x, p)$$

$$\left(p_\alpha{}^i=\frac{\partial x^i}{\partial u^\alpha};\ i, j, k=1, 2, \cdots, N;\ \alpha, \beta, \gamma=1, 2, \cdots, K\right),$$

where $\Pi_{jk}{}^i$ denotes the descriptive connection coefficient. If a subspace S contains all the K-spreads which pass through a point P and are tangent to an L-vector at P $(K<L)$, then S is called a subspace flat at P. A subspace is totally flat if it is flat at all its points. The following postulate is called the axiom of the plane: For a certain integer L $(1<K<L<N)$ there passes a totally flat L-dimensional subspace of class C^2 through each point of the space and tangent to each L-vector at that point. The author proves the following theorem by considering the integrability conditions of the differential equations above: A space of K-spreads is descriptively flat if and only if the axiom of the plane holds.
A. Kawaguchi (Sapporo)

1336:
Su, Buchin. A generalization of descriptive collineations in a space of K-spreads. Math. Nachr. **16** (1957), 227–232.

The author discusses the theory of isomorphic transformations for the descriptive geometry of K-spreads. The associate tensor $\Omega_{jk}{}^i(\xi)$ is introduced as the Lie derivative of the coefficient $\Pi_{jk}{}^i$ of the descriptive connection with respect to an infinitesimal transformation $\bar{x}^i=x^i+\xi^i(x, p)\delta t$, and the vanishing of $\Omega_{jk}{}^i$ is necessary and sufficient for the infinitesimal transformation to be a generalized descriptive collineation. A general mapping of a space of K-spreads onto itself is a generalized descriptive collineation if and only if the descriptive differentiation commutes with the Lie derivation. Further, considering the integrability conditions of $\Omega_{jk}{}^i(\xi)=0$, the conditions are found for the space to admit generalized descriptive collineations. *A. Kawaguchi* (Sapporo)

1337:
Kervaire, Michel A. Sur les formules d'intégration de l'analyse vectorielle. Enseignement Math. (2) **3** (1957), 126–140.

In dieser didaktische Ziele verfolgenden Arbeit wird ein elementarer Beweis für die Integralformeln von Gauss-Ostrogradski und Stokes gegeben. Der Grundgedanke des Beweises besteht darin, dass man Kurven,

Flächen und Rauminhalte zu algebraischen Objekten macht. Es ist nämlich ein jedes dieser Objekte eine endliche ganzzahlige Linearkombination von Elementar-Bogen, -Flächen bzw. Rauminhalten. Die Elementarelemente sind dabei durch eine Parameterdarstellung definiert. Mit den so definierten Objekten kann dann wie mit gewöhnlichen Linearformen gerechnet werden. Der Beweis der Integralformeln beruht darauf, dass man alles auf exakte Weise auf elementare Elemente zurückführen kann. *O. Varga* (Debrecen)

1338:

Malgrange, Bernard. Plongement des variétés analytiques-réelles. Bull. Soc. Math. France **85** (1957), 101–112.

The author proves as follows. If V^n is real analytic (compact or) denumerable at infinity and carries an analytic ds^2, then there exist analytic functions $f_1, \cdots, f_N$ such that $y=f$ is a 1-1 map of V^n into R^N such that the pre-images of compact sets are compact, and locally n among the functions $f_1, \cdots, f_N$ are local coordinates on V^n.

For a compact V^n this had been originally stated and proven by the reviewer by starting out from a Whitney mapping C^r, and approximating to the C^r functions by analytic functions obtainable from the eigenvalue problem $\Delta f=-\lambda f$ with the Laplacian Δf pertaining to ds^2. The author's proof, which is quite elaborate, does not use Whitney's result but proceeds directly, and therefore does not get a bound on N ($N=2n+1$ for compact V^n). Furthermore the eigenvalue problem is replaced by the "equivalent" wave equation $\Delta f+\partial^2 f/\partial t^2=0$; that is, the author forms the product manifold $W=V^n\times R$, where $R=\{-\infty<t<\infty\}$, and he employs harmonic functions on W pertaining to the line element ds^2+dt^2.

An intermediate result, remarkable for itself, is the following lemma. For a manifold Ω let E_Ω be the space of functions C^∞ with the topology of locally uniform convergence for a function and all derivatives. Let Ω_i be a sequence of disjoint open sets in V of which only finitely many meet any compact subset of V. Let f_i be analytic in Ω_i, and u_i a neighborhood of zero in E_{Ω_i}. Then there is a harmonic function on W whose restriction to $V\times(0)$, denoted by $g(x)$, has the property that $g-f_i\in u_i$ for all i simultaneously. *S. Bochner* (Princeton, N.J.)

1339:

Cartan, Henri. Variétés analytiques réelles et variétés analytiques complexes. Bull. Soc. Math. France **85** (1957), 77–99.

In the author's well-known work in the theory of functions of several complex variables, the following two properties of a sheaf of modules S over a sheaf of rings R on a space X have been introduced: A) the images of the natural homomorphisms $H^0(X, S)\to S_x$ generate the R_x-modules S_x at each point $x\in X$; B) $H^q(X, S)=0$ for $q>0$. A fundamental result is that any coherent analytic sheaf on a Stein manifold has properties A and B (the sheaf of rings being the sheaf of germs of holomorphic functions) [Colloque sur les fonctions de plusieurs variables, Bruxelles, 1953; Thone, Liège; Masson, Paris; pp. 41–55; MR **16**, 235]. The author extends these results in the present paper by demonstrating that any coherent analytic sheaf on a closed subset V of a complex manifold has properties A and B, if the subset V possesses a basis of open neighborhoods each of which is a Stein manifold. If in addition the closed subset V is a real analytic manifold which is so imbedded that, in a suitable system of local complex coordinates $z_j=x_j+iy_j$ at any point, V is the subset $\{y_j=0\}$ and the x_j are local coordinates on V, then any real coherent analytic sheaf on V also has properties A and B (the sheaf of rings being the sheaf of germs of real analytic functions); it is demonstrated that the n-dimensional real space R^n imbedded naturally in C^n satisfies these conditions. A real analytic subset V of R^n is called coherent if its sheaf of ideals (defined as in the complex analytic case [loc. cit.]) is a real coherent analytic sheaf; and it is further proved that a real coherent analytic sheaf on a coherent real analytic subset of R^n also has properties A and B. Several consequences of these fundamental results are listed, paralleling those obtained in the complex analytic case [loc. cit.]. The remainder of the paper is devoted to a more detailed study of some related properties of real analytic subsets of R^n. Firstly, conditions are found for a real analytic subset to be coherent (and an example given of a real analytic subset which is not coherent); and secondly, criteria are obtained for a real analytic subset of R^n to be definable globally by a finite number of real analytic functions (and an example given of a real analytic subset which cannot be so defined). *R. C. Gunning* (Princeton, N.J.)

1340:

Malgrange, Bernard. Faisceaux sur des variétés analytiques réelles. Bull. Soc. Math. France **85** (1957), 231–237.

Complementing the two papers reviewed above, the author proves the following two results: firstly, that if a (separable) real-analytic manifold V has the property that $H^1(V, F)=0$ for every real coherent analytic sheaf on V, then V can be realized as an analytic submanifold of R^n; and secondly, that for any real coherent analytic sheaf F on a real analytic manifold V, $H^p(V, F)=0$ for $p\geqq 2$ (or, in the complex analytic case, for $p\geqq k+1$ where k is the complex dimension of V).

R. C. Gunning (Princeton, N.J.)

1341:

Rapcsák, A. Metrische Charakterisierung der Finslerschen Räume mit verschwindender projektiver Krümmung. Acta Sci. Math. Szeged **18** (1957), 192–204.

Finsler spaces of constant and of scalar curvature were defined by Berwald [Ann. of Math. (2) **48** (1947), 755–781; MR **9**, 207], who also introduced the notion of spaces of vanishing projective curvature (loc. cit. p. 762 et seq.), the latter being more general than the former. In the present paper the author introduces certain hypersurfaces by means of which the above-mentioned special types of Finsler spaces may be characterised. In the treatment of these hypersurfaces the element of support is taken in the direction of the unique transversal to the hypersurfaces. Hyperplanes are defined by the condition that the unit transversal vectors should be parallel (with respect to the metric of the imbedding space). Necessary and sufficient conditions for the existence of such hyperplanes are found; these conditions may then be used to distinguish between spaces of zero projective curvature and spaces of constant curvature. Conditions that the metric of the hyperplanes be Riemannian and of constant curvature are also derived. *H. Rund* (Durban)

PROBABILITY

See also 1085, 1123, 1248, 1424, 1592, 1594, 1599.

1342:

***Чеботарев, А. С. Способ наименьших квадратов с основами теории вероятностей. [Čebotarev, A. S. The method of least squares and the foundations of the theory of probability.]** Izdat. Geodez. Lit., Moscow, 1958. 606 pp. (1 insert) 14.50 rubles.

The 28 chapters are grouped into four parts: Theory of the errors of observation; The method of least squares; Supplementary questions on the methods of least squares; The theory of probability. The most interesting feature of the book is the rather extensive treatment of nonlinear least squares, and of problems with side conditions. The methods are known [e.g., Deming, Statistical adjustment of data, Wiley, New York, 1943; MR 5, 208] but not often to be found in the texts.

Illustrative examples are copious and drawn mainly from geodesy. *A. S. Householder* (Oak Ridge, Tenn.)

1343:

Hermes, Hans. Über eine logische Begründung der Wahrscheinlichkeitstheorie. Math.-Phys. Semesterber. **5** (1957), 214–224.

The usual axioms of probability are given an interpretation in terms of betting coefficients. A set of betting coefficients is 'foolish' if it permits an opponent to place bets in such a way as to win in all eventualities (to 'make a book'). If the betting coefficients are not foolish they must satisfy the axioms of probability, and conversely, if they satisfy the axioms of probability then they are not 'foolish'. The argument derives primarily from F. P. Ramsey, "The foundations of mathematics" [Kegan-Paul, London, 1931, pp. 156–198]. It is emphasized that other interpretations of the axioms of probability are possible. *I. J. Good* (Cheltenham)

1344:

Wendel, J. G. Invariance of normal distributions. Michigan Math. J. **4** (1957), 173–174.

A partial converse is given for the well-known assertion that an affine transform of a normal random vector is normal. Let $\mathcal{N}(\mu, \Sigma)$ be an n-variate nonsingular normal distribution with mean vector μ and covariance matrix Σ; let $\mu_1, \mu_2, \cdots, \mu_n$ be n independent vectors in E^n. Then, if g is a borel measurability preserving transformation on E^n which sends the induced distributions $\mathcal{N}(0, \Sigma)$ and $\mathcal{N}(\mu_j, \Sigma)$ into normal distributions ($j=1, 2, \cdots, n$), g is affine almost everywhere. Counter examples are announced for the cases in which the number of transformed normals is less than $n+1$ (with an example for $n=1$). *R. F. Tate* (Seattle, Wash.)

1345:

Medgyessy, Pál. Partial differential equations for stable density functions and their applications. Magyar Tud. Akad. Mat. Kutató Int. Közl. **1** (1956), 489–518 (1957). (Hungarian. Russian and English summaries)

Let

$$f(t)=\exp\{iyt-c|t|^{\alpha}[1+i\beta\frac{|t|}{t}\omega(t,\alpha)]\}$$

(with $0<\alpha\leq 2$, $c\geq 0$, $|\beta|\leq 1$, and $\omega(t,\alpha)=\tan\frac{1}{2}\pi\alpha$ if $\alpha\neq 1$ but $=2\pi^{-1}\log|t|$ if $\alpha=1$) be the characteristic function of a stable distribution and denote by $p(x, c)$ the corresponding density function. The author assumes that α is rational, $\alpha=m/n$ where m, n are integers and $(m, n)=1$, and that $\beta=0$ in case $\alpha=1$. He shows that $p(x, c)$ satisfies a linear partial differential equation with constant coefficients

$$K_1\frac{\partial^{b_1+a_1}p}{\partial c^{b_1}\partial x^{a_1}}+K_2\frac{\partial^{b_2+a_2}p}{\partial c^{b_2}\partial x^{a_2}}+K_3\frac{\partial^{b_3+a_3}p}{\partial c^{b_3}\partial x^{a_3}}=0.$$

Here the a_i, b_i, K_i ($i=1, 2, 3$) depend on m, n and β but are not uniquely determined. This fact is used to simplify the differential equation satisfied by $p(x, c)$. As an application of these results the author studies the decomposition of mixtures of certain stable distributions. *E. Lukacs* (Washington, D.C.)

1346:

Gartštein, B. N. On the limit joint distribution of central range and median of a variational series. Užgorod. Gos. Univ. Naučn. Zap. Him. Fiz. Mat. **12** (1955), 111–127. (Russian)

The distribution described in the title is shown to be normal, with parameters which depend upon the values of the derivative of the original distribution function at the appropriate points. [See also Cramér, "Mathematical methods of statistics", Princeton Univ. Press, 1946; MR 8, 39; Section 28.5.] *J. Wolfowitz* (Ithaca, N.Y.)

1347:

Varadarajan, V. S. Weak convergence of measures on separable metric spaces. Sankhyā **19** (1958), 15–22.

Study of "weak convergence" of measures over a given separable metric space defined by $\int g d\mu_n \to \int g d\mu$ for every bounded continuous function g on this space. The author shows that this convergence in the space M of finite signed measures corresponds to a certain "weak topology" on M, that the subspace M^+ of finite measures and the subspace M_p of probability measures are metrizable as complete metric spaces when the given metric space is complete, and that M is correspondingly metrizable if and only if the "weak" and norm topologies on M are identical. *M. Loève* (Berkeley, Calif.)

1348:

Varadarajan, V. S. On the convergence of sample probability distributions. Sankhyā **19** (1958), 23–26.

The following extension of the Glivenko-Cantelli theorem is proved: "Let $\xi_1, \xi_2, \cdots$ be independent and identically distributed random variables whose values lie in a metric space and let μ be their common distribution on this space. Let μ_n be the sample probability distribution. Then $P(\mu_n \Rightarrow \mu)=1$ if and only if the random variables are strongly measurable. (In any case the probability is 0 or 1.)" *M. Loève* (Berkeley, Calif.)

1349:

***Grenander, Ulf; and Szegö, Gabor. Toeplitz forms and their applications.** California Monographs in Mathematical Sciences. University of California Press, Berkeley-Los Angeles, 1958. vii+245 pp. $6.00.

Various remarkable problems of the modern theory of functions and theory of probability and statistics may be centered about the study of certain hermitian forms whose matrices, finite or infinite, have elements of the form $C_{\nu-\mu}$ and corresponding kernels of two real variables of the form $K(x-y)$. This book unifies the whole subject and frequently completes the known answers. Chapters 1 to 6 and Chapter 9 are due to Szegö, whose contributions to this subject within the theory of functions are classical, and the other chapters are due to Grenander, who

created the statistical theory of stochastic processes with a continuous parameter. Chapter 1 recalls propositions to be used throughout the remainder of the book about integration, monotonic functions, orthogonal systems, approximations, Toeplitz forms, trigonometric moment problem, positive representations, etc. Chapter 2 is devoted to the algebraic properties of orthogonal polynomials, while Chapter 3 is devoted to their limit properties. Chapter 4 presents the trigonometric moment problem. Chapter 5 investigates eigenvalues of Toeplitz forms and Chapters 6 and 7 are concerned with generalizations and analogs of Toeplitz forms, while Chapter 8 is concerned with integral equations of the Toeplitz type. This covers Part I. Part II is devoted to applications: Chapter 9 deals with three problems concerning the inf of $f(z)$ and the sup of $|f(z)|$ (for $|z|<1$) of analytic functions $f(z)$ regular in the unit circle. Chapter 10 deals with the spectral representation and the prediction theory of stochastic processes, while Chapter 11 deals with the estimation of characteristics of stochastic processes. The bringing together of the many problems centered about the same basic instrument is invaluable.

M. Loève (Berkeley, Calif.)

1350:

Helms, Lester L. Mean convergence of martingales. Trans. Amer. Math. Soc. **87** (1958), 439–446.

Let $(\Omega, \mathfrak{B}, \mu)$ be a probability space. If $\mathfrak{B}_0$ is a σ-subalgebra of $\mathfrak{B}$, set $L^p[\mathfrak{B}_0]$ ($p \geqq 1$) as the class of functions that are measurable-$\mathfrak{B}_0$ and whose pth powers are integrable-μ. Let $(A, <)$ be a directed set, and consider the generalized martingale $\{x_a, \mathfrak{B}_a, a \in A, <\}$ in which x_a equals a.e. a function measurable-$\mathfrak{B}_a$, $\mathfrak{B}_a$ is a σ-subalgebra of $\mathfrak{B}$, $\mathfrak{B}_a \subset \mathfrak{B}_b$ if $a<b$, and $E[x_b|\mathfrak{B}_a]=x_a$ a.e. if $a<b$. The author proves the equivalence of the following four propositions when applied to a martingale for which $\{x_a : a \in A\} \subset L^p[\mathfrak{B}]$: ($P_1$) the net $\{x_a : a \in A\}$ is uniformly integrable if $p=1$ or is strongly bounded if $p>1$; (P_2) the net $\{x_a : a \in A\}$ converges weakly in $L^p[\mathfrak{B}]$; (P_3) there exists an $x_\infty \in L^p[\mathfrak{B}]$ such that $\{x_a, \mathfrak{B}_a, a \in A \cup \{\infty\}, <\}$ is a martingale (assume $\mathfrak{B}_\infty=\mathfrak{B}$ and $a<\infty$ for all $a \in A$); (P_4) the net $\{x_a : a \in A\}$ converges strongly in $L^p[\mathfrak{B}]$. This theorem extends and generalizes known results for martingales indexed by the positive integers.

R. Pyke (New York, N.Y.)

1351:

Skorohod, A. V. Limit theorems for stochastic processes with independent increments. Teor. Veroyatnost. i Primenen. **2** (1957), 145–177. (Russian. English summary)

Applying general limit theorems for stochastic processes, obtained in a previous paper [Teor. Veroyatnost. i Primenen. **1** (1956), 289–319; MR **18**, 943], the author now obtains more effective results for the special case in which the processes have independent increments and their sample functions belong to the class K with probability 1. [For terminology, here and below, see the review cited.] In § 2.6 he supposes that $\xi_0(t)$ and $\xi_n(t)$ ($0 \leqq t \leqq 1$) are processes of this type such that $\xi_0(t)$ is stochastically continuous and the finite-dimensional distributions of $\xi_n(t)$ converge as $n \to \infty$ to those of $\xi_0(t)$. He shows that then there exists a (fixed) function $\bar{x}_n(t)$ such that $\bar{x}_n(t) \to 0$ as $n \to \infty$ and that, for every functional f continuous in the J_1-topology, the distribution of $f\{\xi_n(t)-\bar{x}_n(t)\}$ converges to that of $f\{\xi_0(t)\}$. In § 3.4 he supposes that (i) for each n there is given a sequence $\xi_{n1}, \xi_{n2}, \cdots, \xi_{nn}$ of independent random variables and a second such sequence $\eta_{n1}, \eta_{n2}, \cdots, \eta_{nn}$, (ii) $\xi_n(t)=\sum_{i \leq (n+1)t} \xi_{ni}$, $\eta_n(t)=\sum_{i \leq (n+1)t} \eta_{ni}$, (iii) the distributions of $\xi_n(t)$ and $\eta_n(t)$ converge to that of a process $\xi_0(t)$, (iv) f is a functional for which there exists a continuous function $\omega(\delta)$, with $\omega(0)=0$, such that $|f\{x(t)\}-f\{y(t)\}|<\omega(\delta)$ when $\sup_t |x(t)-y(t)|<\delta$. He shows that then the Lévy-distance between the distributions of $\xi_n(t)$ and $\eta_n(t)$ tends to 0 as $n \to \infty$. He obtains in § 4 various limit theorems for functions continuous in other topologies considered in the paper cited, and in § 5 for stochastic processes in a Banach space.

H. P. Mulholland (Exeter)

1352:

Dobrušin, R. L. The continuity condition for sample martingale functions. Teor. Veroyatnost. i Primenen. **3** (1958), 97–98. (Russian. English summary)

It is proved that the condition

$$(1) \qquad \sup_{t \in [0, 1-\Delta t]} P\{|\zeta_t - \zeta_{t+\Delta t}| > \varepsilon\} = o(\Delta t) \text{ for any } \varepsilon > 0$$

is sufficient for almost all sample functions of a separable stochastic process $\{\zeta_t, t \in [0, 1]\}$ not to have first order discontinuities. If the process $\{\zeta_t, t \in [0, 1]\}$ is a martingale process, it is sufficient that almost all sample functions of a separable process are continuous.

Author's summary

1353:

Yaglom, A. M. Certain types of random fields in n-dimensional space similar to stationary stochastic processes. Teor. Veroyatnost. i Primenen. **2** (1957), 292–338. (Russian. English summary)

A thorough investigation of spectral properties of continuous in quadratic mean linear random functionals $\xi(\varphi)$ where the argument φ belongs to the space D of indefinitely differentiable, complex-valued functions of $x \in R_n$, vanishing outside compacts. Spectral representations are obtained when the mean value and the covariance functionals are invariant under (1) the shift transformation in D, (2) the shift transformations in the subspace D_1 determined by $\int \varphi(x)dx=0$, (3) the transformations in D induced by the orthogonal transformations in R_n, and (4) the transformations in D_1 induced by the orthogonal transformations in R_n. The results unify and extend those of Kolmogorov, Cramér, Itô, and the reviewer, and may be expected to play an important role in statistical physics, in particular, in the statistical theory of turbulence and random noise.

M. Loève (Berkeley, Calif.)

1354:

Vinokurov, V. G. Conditions for the regularity of stochastic processes. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 959–961. (Russian)

The author studies stochastic processes with the time parameter running through all the integers. He gives a definition, too lengthy to reproduce here, of regular processes. A theorem then asserts that these are precisely the processes for which the future is arbitrarily little dependent on the remote past, i.e., the processes having the property that to every event A and every $\varepsilon>0$, there corresponds a time t such that if B is an event defined by what happens prior to t then $|P(A \cap B)-P(A)P(B)|<\varepsilon$. Some remarks are made on Markov and on linear processes. It is stated that the results extend to the continuous parameter case. No proofs are given.

A. Dvoretzky (Princeton, N.J.)

1355:

Nagaev, S. V. Some limit theorems for stationary Markov chains. Teor. Veroyatnost. i Primenen. **2** (1957), 389–416. (Russian. English summary)

The author considers stationary Markov chains with

transition probabilities $p(\xi, A)$, $\xi \in X$ and $A \in F_X$ a σ-algebra of subsets of X. It is assumed that for some k

$$\sup_{\eta,\xi\in X,\ A\in F_X} |p^{(k)}(\xi, A)-p^{(k)}(\eta, A)|<1.$$

Let $f(\xi)$ be a real function defined on X. Let $x_1, x_2, \cdots$ be the process obtained for some starting distribution and let $S_n=(1/B_n)\sum_1^n f(x_i)-A_n$, where A_n and B_n are positive constants. It is shown that only stable laws are possible for limit laws for S_n. The central limit theorem is proven under the assumption that $\int_X f^2(\eta)p(d\eta)<\infty$ where $p(\cdot)$ is the stationary measure corresponding to $p(\cdot, \cdot)$. Sufficient conditions for convergence to stable laws are given. Finally, the local limit theorem is proven for denumerable chains and asymptotic expansions are given.

J. L. Snell (Hanover, N.H.)

1356:

Goodman, Leo A. Exact probabilities and asymptotic relationships for some statistics from m-th order Markov chains. Ann. Math. Statist. **29** (1958), 476–490.

The author presents an exact formula for the probability of a specified frequency count of m-tuples in a sequence from a Markov chain of order $m-1$, this being a generalisation of the reviewer's result for $m=2$ [J. Roy. Statist. Soc. Ser. B **17** (1955), 235–242; MR **17**, 982].

The method of proof is the combinatorial one used by Dawson and Good [Ann. Mat. Statist. **28** (1957), 946–956; MR **20**#339]. An exact expression is also obtained for the conditional probability of a specified m-tuple count, given the n-tuple count, when the chain is of order $n-1 (n<m)$. This latter result leads to several simplified tests for hypotheses on the order of the chain. Analogous results are obtained for circular sequences.

P. Whittle (Wellington)

1357:

Fuchs, A. Some limit theorems for nonhomogeneous Markoff processes. Trans. Amer. Math. Soc. **86** (1957), 511–531.

Writing t for non-negative times, a for real numbers, and B for linear Borel sets, if $p(t, a, B)$ is a stationary Markov transition function and if, for a certain time $s\geqq 0$, the measures $p(s, a, \cdot)$ majorize a certain non-negative Borel measure of total mass $1-\alpha>0$, then

$$\lim_{t\to+\infty} p(t, a, B)=p(B) \tag{1}$$

exists for each a and B; in fact,

$$|p(t, a, B)-p(B)|\leqq\alpha^{(t/s)-1} \tag{2}$$

for each $t\geqq s$, a, and B. The proof is similar to W. Doblin's proof of the corresponding fact for chains [see J. L. Doob, Stochastic processes, Wiley, New York, 1953, pp. 197–198; MR **15**, 445]. (2) is also proved for a very narrow class of non-stationary transition functions.

H. P. McKean, Jr. (Cambridge, Mass.)

1358:

Fortet, Robert. Résumés exhaustifs pour un processus de Markov. C. R. Acad. Sci. Paris **247** (1958), 28–29.

Pour certains processus de Markov homogènes, il existe un résumé exhaustif par rapport à l'ensemble des éléments décrivant statistiquement l'évolution.

Résumé de l'auteur

1359:

Getoor, R. K. Additive functionals of a Markov process. Pacific J. Math. **7** (1957), 1577–1591.

Let $L=\int_0^t V[x(\tau)]d\tau$, where $x(t)$ is a temporally homogeneous Markov process in a locally compact Hausdorff space X, and V is a nonnegative measurable function on X. Consider $r(t, x, A)=E\{e^{-uL}|x(0)=x;\ x(t)\in A\}p(t, x, A)$, where $p(t, x, A)$ is the transition probability function of $x(t)$. Under the assumption that $p(\cdot, \cdot, A)$ is jointly measurable in (t, x) for each $A \in \mathcal{B}(X)$, where $\mathcal{B}(X)$ is the class of Borel sets of X, and that V is bounded, $r(t, x, A)$ is shown to satisfy

$$\begin{aligned} r(t, x, A)-p(t, x, A) &= -\int_0^t ds \int V(y)r(t-s, y, A)p(s, x, dy) \\ &= -\int_0^t ds \int V(y)p(t-s, y, A)r(s, x, dy). \end{aligned}$$

$p(t, x, A)$ is then assumed to be absolutely continuous with respect to a measure m on $\mathcal{B}(X)$ for $t>0$. Various assumptions on the derivative $f(t, x, y)$ are introduced. The semigroups T_t, U_t, $t>0$, on $L_2(m)$

$$(T_t\varphi)(x)=\int\varphi(y)r(t, x, dy),$$

$$(U_t\varphi)(x)=\int\varphi(y)f(t, x, y)m(dy)$$

are introduced. Under the assumptions made, the infinitesimal generators Ω, Ω' of U_t, V_t are introduced, and shown to satisfy $\Omega'=\Omega-V$. The results continue previous work on such functionals [see Darling and Siegert, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 525–529; MR **18**, 155; M. Kac, Proc. of second Berkeley Sympos. on Math. Stat. and Probability, U. Calif. Press, Berkeley, 1951; MR **13**, 568; M. Rosenblatt, Trans. Amer. Math. Soc. **71** (1951), 120–135; MR **13**, 258].

M. Rosenblatt (Bloomington, Ind.)

1360:

Breny, H. Sur les "délais de passage" de certaines fonctions aléatoires de Markov. Bull. Soc. Roy. Sci. Liège **27** (1958), 5–16.

Generating functions for passage times of birth and death processes with non-negative integer states.

H. P. McKean, Jr. (Cambridge, Mass.)

1361:

Jiřina, Miloslav. The asymptotic behaviour of branching stochastic processes. Czechoslovak Math. J. **7(82)** (1957), 130–153. (Russian. English summary)

Die Arbeit behandelt das asymptotische Verhalten von n-dimensionalen stationären Markoffschen verzweigten Prozessen im Fall $R<1$, wobei R der größte Eigenwert der Matrix $\mathbf{A}$ der ersten Momente sei. Im Fall diskreter Parameter seien $P_\alpha^\beta(t)$ die Übergangswahrscheinlichkeiten, also α und β Vektoren mit n nichtnegativen ganzzahligen Komponenten. ε_i bedeute den i-ten Einheitsvektor $(0, \cdots, 0, 1, 0, \cdots, 0)$, $F_i(t, x)=\sum_\alpha P_{\varepsilon_i}^\alpha(t)x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ die erzeugenden Funktionen, $A_{ij}(t)=\sum_\alpha \alpha_j P_{\varepsilon_i}^\alpha(t)$ die Matrix der ersten Momente, so daß $\mathbf{A}=(A_{ij}(1))$ wird, μ und ν einen rechten und linken zu R gehörigen Eigenvektor von $\mathbf{A}$ mit $\sum_j \mu_j\nu_j=1$ und $Q_i(t)=1-P_{\varepsilon_i}^0(t)$ die Wahrscheinlichkeit dafür, daß der Prozeß vom Zustand ε_i aus zur Zeit t nicht ausstirbt. Ist nun $\mathbf{A}$ primitiv und irreduzibel und sind auch alle zweiten Momente vorhanden, so existieren die Grenzwerte $K_i=\lim_{t\to\infty}(Q_i(t)/R^t)$, und die bedingten Wahrscheinlichkeitsverteilungen $P_{\varepsilon_i}^\alpha(t)/Q_i(t)$ konvergieren gegen Wahrscheinlichkeitsverteilungen, deren erzeugende Funktionen $F_i^*(x)$ der Funktionalgleichung $F_i^*(F(1, x))=RF_i^*(x)+1-R$ genügen und die ersten Momente $(\mu_i\nu_k)/K_i$ haben. Ein entsprechender Satz gilt im Falle kontinuierlicher Parameter, wobei an die Stelle der Funktionalgleichung eine partielle Differentialgleichung tritt. — Die Theorie der

Verzweigungsprozesse wird schließlich auf den Fall des asymptotischen Gleichgewichts eines Systems von Teilchen angewandt, die alle aus einem System von Teilchen T_0 entstehen, falls die Anzahl der Teilchen T_0 wächst und die Übergangsgeschwindigkeiten klein werden. Unter geeigneten Voraussetzungen wird die Existenz eines asymptotischen Gleichgewichts bewiesen, und die erzeugende Funktion und die Momente der Gleichgewichtsverteilung werden berechnet. *K. Krickeberg* (Hamburg)

1362:

Dobrušin, R. L. Some classes of homogeneous denumerable Markov processes. Teor. Veroyatnost. i Primenen. **2** (1957), 377–380. (Russian. English summary)

If $(p_{ij}(\cdot))$ is the matrix of transition probability functions of a Markov chain with countably many states, the matrix $(p_{ij}'(0))$ is called the transition density matrix of the chain. The author gives a complete characterization of the class of chains having a given transition density matrix, with finite elements and zero row sums, and also having no beginning and finitely many ends. The latter two conditions are defined as follows. A process has no beginning if there is no sequence $\{i_k, k\geqq 1\}$ of distinct integers such that $a_{i_{k+1}i_k}>0$ for all k. A process has n ends $R_1, \cdots, R_n$ if all states can be divided into n disjunct subsets $R_1, \cdots, R_n$ such that, if D_R is the event that a sample path finally reaches and remains in R, then (1) $P(D_{R_i})>0$ for all i; (2) $\sum_i P(D_{R_i})=1$; (3) if $R\subset R_i$, $P(D_R)$ is either 0 or $P(D_{R_i})$. The analysis, only outlined in the paper, is in terms of the character of the sample functions of the processes, by way of such probabilities as the probability of a transition from R_i to R_j at a limit point of jumps. *J. L. Doob* (Urbana, Ill.)

1363:

Karlin, Samuel; and McGregor, James. The classification of birth and death processes. Trans. Amer. Math. Soc. **86** (1957), 366–400.

Let $A\equiv\{a_{ij}:i, j=0, 1, 2, \cdots\}$ be defined by $a_{i,i+1}=\lambda_i$, $a_{i,i-1}=\mu_i$, $a_{i,i}=-(\lambda_i+\mu_i)$, $a_{ij}=0(|i-j|>1)$, where λ_i and μ_i are positive and finite for all i save that (for the moment) $\mu_0=0$. Let us call $P(t)\equiv\{p_{ij}(t):i, j=0, 1, 2, \cdots\}$ $(t\geqq 0)$ a B-system when the conditions

$$(*)\qquad P(0)=I,\ P(t)\geqq 0,\ \sum_j p_{ij}(t)\leqq 1,$$

$$P(t+s)=P(t)P(s),$$

$$P'(t)=AP(t)\ (t\geqq 0),$$

are satisfied elementwise, and a BF-system when in addition $P'(t)=P(t)A$. A B- or BF-system will be called 'honest' when equality holds in the last of the inequalities (*). The corresponding existence and uniqueness problems have been treated by several writers; the most complete result is that given by G. E. H. Reuter [Acta Math. **97** (1957), 1–46]. He defined quantities R and S in terms of the λ's and μ's in such a way that: (i) when $R=\infty$, then there is exactly one B-system; it is a BF-system and it is honest; (ii) when $R<\infty$ but $S=\infty$, then there are infinitely many B-systems, of which exactly one is a BF-system, and it is not honest; (iii) when $R<\infty$ and $S<\infty$, then there are infinitely many BF-systems, of which exactly one is honest.

The birth-and-death processes discussed in the present paper are the BF-systems of classes (i) and (ii); a similar approach to the honest system of class (iii) is much to be desired. Reuter's classification theorem applies without change if $\lambda_0=0$; the authors treat this case by relabelling the states as $-1, 0, 1, \cdots$ and then ignoring the 'absorbing' state -1, so that in their notation we then have $\mu_0>0$. In an earlier paper [Trans. Amer. Math. Soc. **85** (1957), 489–546; MR **19**, 989] they studied the connexion between BF-systems and the Stieltjes moment problem, obtaining a classification like (but not quite as detailed as) (i)–(iii) above, and obtaining for BF-systems of types (i) and (ii) representations of the form

$$(**)\qquad p_{ij}(t)=\pi_j\int_0^\infty e^{-xt}Q_i(x)Q_j(x)d\psi(x)$$

where the π's are constants and the Q's are polynomials depending on the λ's and μ's, and ψ is a positive Borel measure on $[0, \infty)$ with regard to which the Q's are orthogonal. Particular representations like (**) had been given earlier by M. Kac [Amer. Math. Monthly **54** (1947), 369–391; MR **9**, 46] and by W. Ledermann and G. E. H. Reuter [Philos. Trans. Roy. Soc. London Ser. A **246** (1954), 321–369; MR **15**, 625]. The authors here exploit their theory of representations (**) for BF-systems of types (i) and (ii) by finding (in terms of the λ's and μ's) necessary and sufficient conditions for recurrence and for positive-recurrence, moments for the first-passage time distributions, moments (in case (ii)) for the time taken by the process to carry out an infinity of transitions, and moments for the time to absorption (when the lowest state is an absorbing one). They even find the conditional moments for the time to absorption in the case when absorption is possible but not certain, and they study the number of transitions which occur before absorption takes place. Perhaps the most remarkable of the individual results in the paper is their theorem that the ratio $p_{ij}(t)/p_{kl}(t)$ has a finite positive limit when t tends to infinity. The possibility of extending this theorem to classes of Markov processes not of the birth and death type will doubtless attract much attention.

The paper concludes with a study of the problem of computing the measure ψ which occurs in (**), and with the evaluation of numerous integrals with respect to ψ which are relevant to the earlier calculations.

D. G. Kendall (Oxford)

1364:

Dvoretzky, A.; Erdös, P.; Kakutani, S.; and Taylor, S. J. Triple points of Brownian paths in 3-space. Proc. Cambridge Philos. Soc. **53** (1957), 856–862.

Es war bekannt, daß die Trajektorien der Brownschen Bewegung in R_2 für jedes k mit $k=2, 3, \cdots$ fast sicher k-fache Punkte haben, daß sie in R_3 fast sicher Doppelpunkte besitzen und daß es in R_n mit $n\geqq 4$ fast sicher keine Doppelpunkte gibt. Die Verf. zeigen nun, daß in R_3 auf den Trajektorien fast sicher keine Tripelpunkte liegen. Hierzu beweisen sie, daß die Menge D der Doppelpunkte fast sicher die Vereinigung abzählbar vieler kompakter Mengen endlichen linearen Maßes ist, die demnach die Kapazität Null haben, so daß D nach einem Satz von Kakutani fast sicher nicht ein drittes Mal getroffen wird. Die Verf. vermuten, daß D fast sicher die Hausdorffsche Dimension 1 hat und daß in der Ebene die Menge der k-fachen Punkte fast sicher die Dimension 2 besitzt.

K. Krickeberg (Heidelberg)

1365:

Fagen, R. E.; and Lehrer, T. A. Random walks with restraining barrier as applied to the biased binary counter. J. Soc. Indust. Appl. Math. **6** (1958), 1–14.

The type of biased binary counter discussed in this paper is one which is restricted to non-negative numbers

less than N, and which advances r units on receipt of a 1 and $-l$ units on receipt of a 0. The condition $pr-ql<0$ where $p=1-q=$prob. of a 1, produces a tendency to drift toward the origin. Such a counter is useful as it can produce an alarm when a process is "out of control", in this case too many 1's.

The paper gives the mean duration of the corresponding random walk starting at the origin, as well as related information on the variance of the duration. A numerical example is given to illustrate the use of the results.

R. W. Hamming (Murray Hill, N.J.)

1366:

Čencov, N. N. Le mouvement brownien à plusieurs paramètres de M. Lévy et le bruit blanc généralisé. Teor. Veroyatnost. i Primenen. **2** (1957), 281–282. (Russian. French summary)

For a class V of lines in the plane, let $\mu(V)$ be the measure of V invariant under rigid motions, normalized so that, if V is the class of lines intersecting a line segment, $\mu(V)$ is the length of the line segment. For each measurable V of finite measure, let $\xi(V)$ be a random variable which is normally distributed with mean 0 and variance $\mu(V)$, and suppose that the stochastic process $\{\xi(V), \mu(V)<\infty\}$ is normal (Gaussian). Let O be some point of the plane and, for each point A, let $\eta(A)$ be $\xi(V)$, where V here is the class of lines meeting the segment OA. The η process is then Lévy's Brownian motion process with plane parameter [Processus stochastiques et mouvement brownien, Gauthier-Villars, Paris, 1948; MR **10**, 551]. There is a corresponding result in Euclidean space of any dimensionality.

J. L. Doob (Urbana, Ill.)

1367:

Zitek, František. Courants d'entrée singuliers. Časopis Pěst. Mat. **83** (1958), 41–55. (Czech. Russian and French summaries)

The title of this work designates stochastic processes $x(t)$ which can reasonably be called arrival processes, that is, they represent customers joining a queue, or units arriving at some service mechanism. Almost surely, $x(t)$ is real, positive, constant except for jumps, and nondecreasing; $x(t)$ is singular if a.s. jumps can only occur at (essentially) integer points in time. Although a genuine need for an adequate theory of such processes has existed in congestion theory for the past twenty years, very little has appeared recently except for Hinčin's monograph [Mathematical methods of the theory of mass service, Trudy Mat. Inst. Steklov. v. 49. Izdat. Akad. Nauk SSSR, Moscow, 1955; MR **17**, 276]. Zítek's paper proves analogues of some results of Hinčin, and discusses some very elementary examples in terms of Markov chains.

V. E. Beneš (Murray Hill, N.J.)

1368:

Takács, Lajos. On stochastic processes which occur in the theory of particle counters. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **6** (1956), 369–421. (Hungarian)

This is a detailed exposition, with an extensive bibliography, of the theory of counters. Let $\{t_n\}$ be the times at which events of the primary process (arrival of particles at the counter) occur and suppose that the distribution of inter-arrival times $t_{n+1}-t_n$ is known. Assume further that the filter effect of the counter produces a rarefied sequence $\{t_n'\}$ of instants at which events of the secondary process (arrival of particles registered) occur. Let ν_t be the number of counts during the period $(0, t)$; the distribution of ν_t is studied for various counter models such as type I and type II counters and their generalizations suggested by G. E. Albert and L. Nelson [Ann. Math. Statist. **24** (1953), 9–22; MR **14**, 775]. Coincidence phenomena which occur if several counters are used simultaneously are also discussed.

E. Lukacs (Washington, D.C.)

1369:

Takács, Lajos. On a certain waiting time problem. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **7** (1957), 183–197. (Hungarian)

This is the Hungarian version of the author's paper in Teor. Veroyatnost. i. Primenen. **2** (1957), 92–105 [MR **19**, 692].

E. Lukacs (Washington, D.C.)

1370:

Rényi, Alfréd. A characterization of Poisson processes. Magyar Tud. Akad. Mat. Kutató Int. Közl. **1** (1956), 519–527 (1957). (Hungarian. Russian and English summaries)

A recurrent process is formed by a sequence of events which occur at random times $0=t_0<t_1<\cdots<t_n<\cdots$ and where the time intervals t_n-t_{n-1} between consecutive occurences are independently and identically distributed random variables with common distribution function $F(x)$. The author considers a transformation T_q of this process which consists of a contraction (=replacement of the time point t_n by qt_n with $0<q<1$) followed by a rarefaction (=cancellation of events, independently of each other, with probability $1-q$). The application of T_q transforms the given recurrent process into a new recurrent process; the distribution function of the time intervals between consecutive occurrences in the transformed process is denoted by $T_q[F(x)]$. The author's principal result is a characterization of the Poisson process (i.e., the process with $F(x)=1-e^{-\theta x}$): the Poisson process is the only process which is invariant under the transformation T_q. It is also shown that $\lim_{q\to 0} T_q[F(x)] = 1-e^{-\theta x}$ provided that (i) $F(0)=0$ and (ii) the first moment of $F(x)$ is finite. Moreover, if (i) and (ii) are satisfied and if the sequence q_j is such that $0<q_j<1$, $\lim_{n\to\infty} q_1q_2\cdots q_n=0$, then the recurrent process obtained by applying consecutively the transformations T_{q_1}, T_{q_2}, $\cdots$, T_{q_n} approaches a Poisson process.

E. Lukacs (Washington, D.C.)

1371:

Miller, K. S.; Bernstein, R. I.; and Blumenson, L. E. Rayleigh processes. Quart. Appl. Math. **16** (1958), 137–145.

Let $x_i(t)$ $(i=1, 2, \cdots, N)$ be independent stationary Gaussian processes with means a_i and the common autocovariance function $\psi(\tau)$. {The authors incorrectly call it an autocorrelation function.} A generalized Rayleigh process, $y(t)$, is defined by $y^2(t)=\sum_{i=1}^{N} x_i^2(t)$. The authors derive the first and second order probability distributions for $y(t)$, in particular the autocovariance function, and in the case $a_i=0$, the third order probability distribution. The first is, of course, the non-central χ distribution when $\psi(0)=1$.

D. V. Lindley (Cambridge, England)

1372:

Lamperti, John. An occupation time theorem for a class of stochastic processes. Trans. Amer. Math. Soc. **88** (1958), 380–387.

Let X_n be a stochastic process whose state space is divided into A, B, and the one-point set $\{\sigma\}$. Assume that the process can get from A to B only by passing through σ. Assume that σ is a recurrent event, f_n being the probability of recurrence in n steps, and F being the generating

function $F(x)=\sum_{n=1}^{\infty} f_n x^n$. Let N_n be the occupation time up to n of the set A (counting σ or not, according to whether the last other state occupied was in A).

Theorem: (a) $\lim_{n\to\infty} \Pr\{N_n/n\leq t\}=G(t)$ exists for all t if and only if $\lim_{n\to\infty} E\{N_n/n\}$ exists, call it α, and $\lim_{x\to 1}(1-x)F'(x)/(1-F(x))$ exists, call it δ. (b) If these limits exist, and $0<\delta<1$, then $G=G_{\alpha,\delta}$ is the distribution function with density

$$G'_{\alpha,\delta}(t)=\frac{a\sin\pi\delta}{\pi}\,\frac{t^{\delta}(1-t)^{\delta-1}+t^{\delta-1}(1-t)^{\delta}}{a^2t^{2\delta}+2at^{\delta}(1-t)^{\delta}\cos\pi\delta+(1-t)^{2\delta}},$$

where $a=(1-\alpha)/\alpha$. $G_{0,\delta}(t)=1$, $G_{1,\delta}(t)=0$ for $0<t<1$; and $G_{\alpha,1}(t)=0$ for $t<\alpha$ and $=1$ for $t\geqq\alpha$, $G_{0,\alpha}(t)=1-\alpha$ for $0\leq t<1$.

The general idea of the proof is like that used by Spitzer in considering the case of sums of independent identically distributed random variables. The class of processes to which the theorem applies contains, in particular, all Markov chains for which one state separates the rest in the manner described above; for example, the case of 1-dimensional random walk.

J. Feldman (Princeton, N.J.)

1373:

Silverman, Richard A. On radio scattering by dielectric turbulence. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-98** (1956), i+30 pp.

The physical theory of the scattering of electromagnetic waves by turbulent atmospheric fluctuations ("dielectric noise") developed by Villars and Weisskopf [Phys. Rev. (2) **94** (1954), 232–240; MR **15**, 762; also Silverman and Balser, ibid. (2) **96** (1954), 560–563; MR **16**, 776] is formalized in terms of the concepts of the theory of random functions and the language of measure theory. It is shown that the amplitude of the scattered signal received is a complex gaussian random function of time with mean zero and that the statistical structure is therefore specified by its covariance.

S. Chandrasekhar (Williams Bay, Wis.)

STATISTICS

See also 1176, 1349, 1400.

1374:

***Neyman, Jerzy. Current problems of mathematical statistics.** Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, Vol. 1, pp. 349–370. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam; 1957. 582 pp. $7.00.

In this paper the author presents a brief review of some of the problems in mathematical statistics which have been investigated in recent years. In particular he discusses stochastic models of clustering; problems of distinguishing between stochastic models leading to observable random variables having identical distribution; problems in statistical decision theory; recent results in the theory of maximum likelihood estimation, sufficient statistics and statistical hypothesis testing. The author did not undertake to include a description of recent results in such topics as non-parametric statistical inference, multivariate statistics, time series, and multiple comparisons. *S. S. Wilks* (Princeton, N.J.)

1375:

Steinhaus, Hugo. On some fundamental questions in mathematical statistics. Pokroky Mat. Fys. Astr. **2** (1957), 36–43. (Czech)

A translation from the German of the article in Bericht Tagung Wahrschein. Math. Statist. Berlin, 1954, pp. 55–63 [Deutscher Verlag der Wissenschaft., Berlin, 1956; MR **19**, 73].

1376:

***Dixon, Wilfrid J.; and Massey, Frank J., Jr. Introduction to statistical analysis. 2nd ed.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1957. xiii+488 pp. $6.00.

This is a textbook in elementary statistics in which no knowledge of calculus is assumed but which provides a comprehensive treatment within the limits thus imposed. Chapter headings include: distributions; measures of central value and dispersion; universe and sample; the normal distribution; statistical inference; estimation and tests of hypotheses; analysis of variance; regression and correlation; enumeration statistics; probability of accepting a false hypothesis, macrostatistics; microstatistics; nonparametric statistics; sequential analysis; sensitivity experiments; probability. A notable feature is the inclusion of a wide selection of statistical tables in the appendix. The principal changes from the first edition [McGraw-Hill, New York, 1951] are partial rewriting of Chapters 6 and 7 on central value and dispersion and statistical inference, complete rewriting of chapter 10 on analysis of variance, and the inclusion of a new chapter on probability. *D. C. Murdoch* (Vancouver, B.C.)

1377:

Des Raj; and Khamis, Salem H. Some remarks on sampling with replacement. Ann. Math. Statist. **29** (1958), 550–557.

In sampling with replacement from a finite population one may (A) make a fixed number of selections, regardless of repetitions, or (more commonly) one may (B) continue drawing until a preassigned number of distinct sampling units have been chosen. It is shown that in either case the mean of the "distinct" units, $\bar{y}_1$, is a more precise estimator of the population mean than is the mean of all selections, including repetitions, $\bar{y}_2$. The proof is somewhat lengthy and the result can be demonstrated more simply by observing that in either case $E(\bar{y}_2|\bar{y}_1)=\bar{y}_1$ so that $\mathrm{Var}\,(\bar{y}_2)=\mathrm{Var}\,(\bar{y}_1)+E(\bar{y}_2-\bar{y}_1)^2\geqq\mathrm{Var}\,(\bar{y}_1)$. The claimed extensions to ratio estimators make uncritical use of certain approximations and these results, although plausible, cannot be considered proved.

P. Meier (Baltimore, Md.)

1378:

Murthy, M. N. Ordered and unordered estimators in sampling without replacement. Sankhyā **18** (1957), 379–390.

Several sampling schemes have been proposed wherein the estimates depend on the order of the draw. In such cases the Rao-Blackwellized estimate (i.e., the conditional expectation of the original estimate given the unordered sample) has smaller variance than the original estimate. *J. L. Hodges, Jr.* (Berkeley, Calif.)

1379:

Rider, Paul R. Generalized Cauchy distributions. Ann. Inst. Statist. Math. Tokyo **9** (1958), 215–223.

The author considers the family of d.f. $f(x)=c(1+$

$|x-\theta|^k)^{-h}$, $-\infty<x<\infty$. The case $h=1$, $k=2$ is the well known Cauchy d.f. and has been studied in some detail. For various other values of h and k, the efficiency of the mean and median as estimators of θ are compared.

I. Olkin (Stanford, Calif.)

1380:

Goodman, Leo A.; and Hartley, H. O. The precision of unbiased ratio-type estimators. J. Amer. Statist. Assoc. **53** (1958), 491–508.

When each unit in a population of N units consists of an x and a y measurement, where the population mean $\bar{X}$ of the x's is known, then the population mean $\bar{Y}$ of the y's may be estimated by drawing a random sample of n units $(x_i y_i)$ $(i=1, \cdots, n)$ and using a ratio estimator

$$\bar{y}=\bar{X}\frac{\bar{y}}{\bar{x}} \text{ or } \bar{y}=\bar{X}\bar{r},$$

where $\bar{x}$, $\bar{y}$, $\bar{r}$ are the means of the x_i, y_i and $r_i=y_i/x_i$ respectively. These are known to be biased estimators in general. The authors introduce a new estimator

$$y'=\bar{X}\bar{r}+\frac{(N-1)n}{N(n-1)}(\bar{y}-\bar{r}\bar{x})$$

and show this is unbiased.

Making use of the variable r_i and its population mean $\bar{R}$, exact formulae are given for variances of these estimators (for $n<<N$), and conditions set down (on the population moments) under which var $y'<$var $\hat{\bar{y}}$ or var $\bar{y}$. A number of special cases are also discussed as well as some regression estimators.

D. G. Chapman (Raleigh, N.C.)

1381:

Tukey, John W. A problem of Berkson, and minimum variance orderly estimators. Ann. Math. Statist. **29** (1958), 588–592.

The (symmetrical) double exponential distribution with known scale and unknown location parameter, is an example where the maximum likelihood estimator while asymptotically efficient is not a minimum variance estimate for any finite sample size. The author shows that the technique used to show this can be extended to yield some interesting results concerning a class of distributions with suitable monotonicity properties. The distributions $F(z)$ considered are called subexponential to the right and are characterized by the property that

$$\frac{F(z+h)-F(h)}{1-F(h)}$$

is a monotonically decreasing function of h for $z>0$. Among the results proved is the theorem that if a sample of n observations is drawn from such a subexponential distribution function $F(z)$, then the covariances of the order statistics of this sample are monotone in either index separately.

D. G. Chapman (Raleigh, N.C.)

1382:

Aitchison, J.; and Silvey, S. D. Maximum-likelihood estimation of parameters subject to restraints. Ann. Math. Statist. **29** (1958), 813–828.

Given a random sample of n from a population whose probability density functions, $f(x, \theta)$, is a function of s parameters (θ) which in turn satisfy r functional relationships $h_j(\theta)=0$. Under fairly general conditions, it is shown that the maximum likelihood estimators of θ are asymptotically jointly normally distributed. Let $l_k(x, \theta)=\partial L(x, \theta)/\partial\theta_k$ and $H(\theta)$ be the $s\times r$ matrix $[\partial h_j(\theta)/\partial\theta_k]$, where $L(x, \theta)=\sum_{i=1}^{n}\log f(x_i, \theta)$.

The ML estimators are found by solving for $\hat{\theta}(x)$ and $\hat{\lambda}(x)$ in the equations

$$l(x, \hat{\theta})+\hat{\lambda}H(\hat{\theta})=0,$$
$$h(\hat{\theta})=0.$$

The asymptotic variances and covariances are given for $n^{\frac{1}{2}}\hat{\theta}$ and $n^{\frac{1}{2}}\hat{\lambda}$; also an iterative procedure for solving the M.L. equations and estimating the variances and covariances.

Asymptotic χ^2-tests are introduced to test (i) that the true parameter point θ_0 satisfies $h(\theta_0)=0$ and (ii) that the true parameter point is some hypothetical value, θ^*. The first test is based on r degrees of freedom and the second on $s-r$.

R. L. Anderson (Raleigh, N.C.)

1383:

Thionet, Pierre. Sur les pertes d'information qui sont des fonctions de risque. C. R. Acad. Sci. Paris **246** (1958), 367–369.

Continuing the investigations of an earlier paper [C. R. Acad. Sci. Paris **246** (1958), 46–49; MR **19**, 1095] the author finds necessary conditions for an expected loss to be a loss of information in the sense previously defined by him.

D. V. Lindley (Cambridge, England)

1384:

Thionet, Pierre. Sur les pertes d'information imputables à certaines estimations biaisées. C. R. Acad. Sci. Paris **246** (1958), 536–539.

The theory of the paper reviewed above is applied to the case of biased estimates, in particular to the ratio of two estimates.

D. V. Lindley (Cambridge, England)

1385:

Thionet, Pierre. Sur une théorie générale des pertes d'information par sondage. C. R. Acad. Sci. Paris **246** (1958), 692–694.

The author extends his investigations into information loss in the papers reviewed above, and extends the theory to more general sampling using the abstract language of lattice theory.

D. V. Lindley (Cambridge, England)

1386:

Rubin, Herman. Estimation of a regression line with both variables subject to error under an unusual identification condition. Ann. Math. Statist. **29** (1958), 606–608.

Let $x=\xi\cos\theta+u$, $y=\xi\sin\theta+v$, where θ is a constant, ξ, u, and v are independent chance variables, the moment generating function (m.g.f.) of (u, v) exists in a neighborhood of the origin, and the m.g.f. of ξ does not exist for all points in any neighborhood of the origin. The author refers to an unpublished paper for the fact that the distribution function of (x, y) uniquely determines θ (mod π). (A trivial proof consists simply of noting that θ is the only value of φ (mod π) for which the m.g.f. of $s(\varphi)=x\sin\varphi-y\cos\varphi=\xi\sin(\varphi-\theta)+u\sin\varphi-v\cos\varphi$ exists in some neighborhood of the origin.) Let $\mu_{2n}(\varphi)$ and $m_{2n}{}^N(\varphi)$ be, respectively, the $(2n)$th moment of $s(\varphi)$ and the $(2n)$th sample moment of $s(\varphi)$ from N independent observations. Since θ is the only value of φ (mod π) for which $\max_n[\mu_{2n}(\varphi)]^{1/2n}(n)^{-1}$ is finite, one shows readily that the value of φ which minimizes $\max_n[m_{2n}{}^N(\varphi)]^{1/2n}(n)^{-1}$ converges to θ with probability one. (The above statements about the moments and m.g.f. of $s(\varphi)$ obviously hold when u and v are not independent. Hence the requirement that u and v be independent is completely unnecessary.)

J. Wolfowitz (Ithaca, N.Y.)

1387:

Schmetterer, L. Bemerkungen zur Theorie der erwartungstreuen Schätzfunktionen. Mitteilungsbl. Math. Statist. **9** (1957), 147–152.

Let us consider a class K of distribution-functions $F(x)$, with density W_F, and a functional $g(F)$ defined on K. The function $h(x)$ is a consistent estimation of $g(F)$ if, for any $F \in K$,

$$E[h(x); F] = \int_{-\infty}^{+\infty} h(x)dF = g(F).$$

E. L. Lehmann and H. Scheffe [Sankhyā **10** (1950), 305–340; MR **12**, 511] proved that, if the dispersion $\Delta = E[(h(x)-g(F))^2; F]$ exists, then there exists an $h(x)$ for which Δ is a minimum. As a result of this fact, $h(x)$ is an element of a strictly convex Banach-space, having the norm $\sqrt{E[(h(x)-g(F))^2]}$.

The author next considers abstract absolute moments of order p, when $v(x)$ belongs to class H_0^p if $E(v(x), F)=0$, $F \in K$ and $E(|v(x)|^p, F)<\infty$. The following general theorem is also proved: $h(x)$ is a consistent estimation for $g(F)$ and renders minimum, for at least one value of p, the expression $E[(h(x)-g(F))^p]$ when $v(x) \in H_0^p$ and

$$E[|h(x)-g(F)|^{p/q} \operatorname{sign}((h(x)-g(F))v(x))]=0,$$

where $p^{-1}+q^{-1}=1$, $q>0$. *O. Onicescu* (Bucharest)

1388:

Fernandes Costa, M. A. Note on the problem of comparison of the means of two normal universes. Gaz. Mat., Lisboa **17** (1956), no. 63–64, 9–11. (Portuguese)

Expository remarks on the Behrens-Fisher problem. *H. L. Seal* (New York, N.Y.)

1389:

Afriat, S. N. Orthogonal and oblique projectors and the characteristics of pairs of vector spaces. Proc. Cambridge Philos. Soc. **53** (1957), 800–816.

It is well known that many concepts in multivariate analysis are equivalent to operations on a space spanned by the vectors of observations, e.g., multiple and partial linear regression can be represented as projections. With the statistical concepts as motivation, the author develops and unifies the requisite algebraic theory. The applications are promised in a later paper.

I. Olkin (Stanford, Calif.)

1390:

Clark, Charles E.; and Williams, G. Trevor. Distributions of the members of an ordered sample. Ann. Math. Statist. **29** (1958), 862–870.

Approximations for the moments of order statistics are obtained by applying a Taylor expansion to the inverse of the cumulative distribution function. The method and the results are closely related to those of F. N. David and N. L. Johnson [Biometrika **41** (1954), 228–240; MR **16**, 382]. *W. Hoeffding* (Chapel Hill, N.C.)

1391:

De Munter, Paul. Efficacités du test de Brown-Mood relatif à la régression. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 838–846.

For the case of observations on k different populations, there is a usual "analysis of variance" test of the hypothesis that the k population means lie on a specified line. This test is appropriate for normal observations satisfying the usual requirements for analysis of variance. In the absence of normality, there is a "distribution free" test for the analogous problem (with population medians playing the role of means), which the author attributes to Brown-Mood. The large sample efficiency (in the sense of Pitman) of these two procedures are compared for alternatives which specify straight lines "close" to the hypothesized one. Some required conditions such as independence, or weaker substitutes for independence are not stated. *M. Dwass* (Evanston, Ill.)

1392:

Guest, P. G. The spacing of observations in polynomial regression. Ann. Math. Statist. **29** (1958), 294–299.

It is desired to fit a polynomial of specified degree p, using n observations on $(p+1)$ points with the independent variable in the interval $(-1, +1)$. The author compares the common procedure of uniform spacing with minimax spacing, i.e. spacing so arranged that the maximum variance of the fitted values in the range $(-1, +1)$ would be minimized. Formulas for the coefficients of σ^2/n for the two variances are given for large n and curves drawn for $p=2$ and 3. Over most of the range of $|x|$, the uniform spacing is slightly better; however, minimax spacing is far superior for $|x|$ near unity and for extrapolated values of x. *R. L. Anderson* (Raleigh, N.C.)

1393:

Beneš, Milan; and Likeš, Jiří. Factorial experiments in industry. I, II. Pokroky Mat. Fys. Astr. **2** (1957), 18–30, 156–166. (Czech)

The authors treat the theory of factorial experiments, particularly with regard to industrial applications. In the first part of the paper basic concepts are introduced and the role of factorial arrangement appraised. General theory for type 2^n experiments is given, the testing procedure described and problems concerning the interpretation of high-order-interactions discussed. Further, several types of 3^n experiments and a method for the decomposition of main effects and interactions are presented. Confounding is explained for the case of a 2^n experiment decomposed in 2^k blocks; the explanation of confounding for 3^n experiments follows. The discussion of fractional factorial experiments is then based on the principles of confounding. Analysis of those experiments is treated briefly since it is based essentially on certain applications of variance analysis. A list of references is given that includes papers on the applications of factorial experiments in some particular fields. *J. Janko* (Prague)

1394:

Beneš, Milan; and Likeš, Jiří. Investigations and determinations of optimal technological procedures. I, II. Pokroky Mat. Fys. Astr. **2** (1957), 523–533, 657–668. (Czech)

Two important features distinguishing industrial experiments from agricultural ones are mentioned in the introduction, viz. sequential arrangement and search for optimum levels of factors. The historical development of the techniques for the determination of optimum levels is then briefly reviewed. Stochastic approximation methods are mentioned in this connection. Box's method of arranging experiments in the search for optimum conditions is then described. The authors explain conditions under which the classical method of changing factor levels is valid. The method of steepest ascent is justified and its application demonstrated by an example involving a type 2^n experiment. Composite designs, proposed by P. Box for the case when a polynomial of second degree must be fitted, are then discussed. The investigation of response

surface in the near stationary region is explained at the end of the paper. *J. Janko* (Prague)

1395:

Pardubský, B. Methods of mathematical statistics for quality control in mass production. I, II. Pokroky Mat. Fys. Astr. **2** (1957), 534–544, 668–674. (Czech)

Expository paper. Fundamental theory underlying control charts for mean and range and for extreme values. *J. Janko* (Prague)

1396:

Cartwright, D. E. On estimating the mean energy of sea waves from the highest waves in a record. Proc. Roy. Soc. London. Ser. A **247** (1958), 22–28.

The author considers a sample of N independent observations from the population with distribution function

$$1-(2\pi)^{-\frac{1}{2}}\left[\int_{x/\varepsilon}^{\infty} e^{-\frac{1}{2}t^2}dt+(1-\varepsilon^2)^{\frac{1}{2}}e^{-\frac{1}{2}x^2}\int_{-\infty}^{x(1-\varepsilon^2)^{1/2}/\varepsilon} e^{-\frac{1}{2}t^2}dt\right]$$

(said to describe the height of sea-waves), and is particularly interested in the distribution of the greatest observation. The significance points and first two moments of this latter distribution are tabulated for a number of values of N and ε, and an asymptotic series (valid for large N) derived for the moments. Some results are also obtained for the second and third largest values, and an assessment made of the effect of serial correlation. A comparison of these results with actual seawave records shows satisfactory agreement.

P. Whittle (Wellington)

1397:

Harybin, A. E. Analysis of errors at determining mean value of the random magnitude and its mean-square error due to finite time of observation. Avtomat. i Telemeh. **18** (1957), 304–314. (Russian. English summary)

$z(t)$ is a stationary process with mean m and correlation function $R(t)$. Let $m(T)=T^{-1}\int_0^T z(t)dt$. The problem considered is that of determining T, the finite time interval of observation subject to the condition that the "relative mean square error" $\sigma=m^{-1}\{E[m(T)]^2-m^2\}^{\frac{1}{2}}$ does not exceed some value, σ_{max}.

Nomographic charts are used "to estimate whether the interval of observation of the random magnitude is sufficient. They also permit to find supposed mean square errors at determining the mean value of the random magnitude for the most typical case when the correlation function of the random process is $R(t)=Ce^{-\alpha|t|}\cos\beta t$." (From the author's summary.)

G. Kallianpur (East Lansing, Mich.)

1398:

***Длин, А. М. [Dlin, A. M.] Математическая статистика в технике. [Mathematical statistics in technical applications.]** 2nd ed. revised. Gosudarstv. Izdat. Sovetsk. Nauka, Moscow, 1951. 292 pp. 8 rubles.

Introduction. Mean indexes of variation. Law of large numbers. Distribution curves. Processing of experimental data. Sampling method. Analysis of industrial errors by the methods of mathematical statistics. Statistical quality control during manufacture. Statistical quality control of products by a qualitative criterion. Current analysis of technico-economic indexes of use of equipment. Dispersion analysis. Application of correlation analysis.

From the table of contents

1399:

Cox, D. R. Some problems connected with statistical inference. Ann. Math. Statist. **29** (1958), 357–372.

The following points are discussed: (i) the differences between decision and inference problems; (ii) the appropriate sample space to be used in making the usual 'tail-area' statements of significance of confidence; (iii) the aim of interval estimation; (iv) the aim of significance tests and the use of 'tail-areas'; and (v) the role of assumptions about the statistical population. In (ii) an example is given of a test which is not admissible but intuitively seems more sensible than any admissible one. But perhaps the most interesting things in this thought-provoking paper are a definition of an ancillary statistic (if (t, a) are jointly sufficient for θ, and $p(t, a|\theta)=p(t|a,\theta)p(a)$, then a is ancillary) and an extension (too long to be given here) to the nuisance parameter situation. The latter involves an attempt at saying what is meant by the statement that an observation gives no direct information about θ. *D. V. Lindley* (Cambridge, England)

NUMERICAL METHODS

See also 1095, 1101, 1122, 1274, 1342, 1431, 1434, 1435, 1436, 1460.

1400:

Halton, J. H.; and Handscomb, D. C. A method for increasing the efficiency of Monte Carlo integration. J. Assoc. Comput. Mach. **4** (1957), 329–340.

An estimator is required for $\theta=\int_0^1 f(x)dx$. Hammersley and Morton [Proc. Cambridge Philos. Soc. **52** (1956), 449–475; MR **18**, 336] used the unbiased estimator $\mathfrak{U}_n f_0(\xi)=n^{-1}\sum_{j=0}^{n-1} f_0((\xi+j)/n)$ where ξ is a random variate uniformly distributed in the range $(0, 1)$ and $f_0(\xi)$ is an unbiased estimator of θ: $\theta=\int_0^1 f_0(x)dx$. Assuming sufficient differentiability, they showed that

$$\mathrm{var}[\mathfrak{U}_n f_0(\xi)]=\frac{\Delta_0^2}{12n^2}+\frac{\Delta_1^2-2\Delta_0\Delta_2}{720n^4}+\cdots,$$

where $\Delta_j=f_0^{(j)}(1)-f_0^{(j)}(0)$ and the remainder after m terms is $o(n^{-2m})$. The present paper constructs estimators $f_0(\xi)$ with the property that $\Delta_m=0$ for $0\leq m\leq M$ and consequently for which $\mathrm{var}[\mathfrak{U}_n f_0(\xi)]=O(n^{-2(M+1)})$. These estimators are certain elementary transformations of f. Explicit formulas are given for $1\leq M\leq 9$. There are discussions of singular integrands and multiple integrals, and there are some worked examples.

P. Davis (Washington, D.C.)

1401:

Lotkin, Mark. Experiments in the smoothing of data. Quart. Appl. Math. **16** (1958), 169–173.

The smoothing of a function $f(x, y)$ where both x and y may be subject to random error, may be treated by the following methods: (i) smooth the function values f only; (ii) smooth the arguments x, y, and then calculate $f(x, y)$ using the smoothed values of x and y; and (iii) subsequently smooth these values of $f(x, y)$.

The author constructs artificial examples with random errors deliberately introduced but nowhere exceeding 1% of the values of x and y, and smooths these using a polynomial process. His conclusions are that method (i) will probably suffice, but if derivatives are required then (ii) should be used. *D. C. Gilles* (Glasgow)

1402:

Spitzbart, A.; and Shell, D. L. A Chebycheff fitting criterion. J. Assoc. Comput. Mach. **5** (1958), 22–31.

If the Chebycheff expansion for a function $f(x)$ is written as

$$f(x)=\sum_{i=1}^{p} a_i x^i+\varepsilon T_{p+1}(x),$$

with $f(x)$ given at the set of $p+2$ points

$$x_k=\sin^2\frac{k\pi}{2(p+1)} \quad (k=0, 1, 2, \cdots, p+1)$$

and A and F are the vectors of the coefficients a_i (including ε) and function values f_k, respectively, then the equation to determine the a_i's may be written as

$$N_p A=F,$$

where N_p is a matrix depending only on p. The authors investigate the form of the inverse matrix N_p^{-1}, and give tables for $p=1(1)5$. The method is also extended to two and three variables. *D. C. Gilles* (Glasgow)

1403:

Ford, L. R., Jr.; and Fulkerson, D. R. Constructing maximal dynamic flows from static flows. Operations Res. **6** (1958), 419–433.

A network, in which two integers t_{ij} (the traversal time) and c_{ij} (the capacity) are associated with each arc P_iP_j, is considered with respect to the following question: what is the maximal amount of goods that can be transported from one node to another in a given number T of time periods, and how does one ship in order to achieve this maximum? A computationally efficient algorithm for solving this dynamic linear-programming problem is given. One especially interesting feature of the algorithm is the fact that the constructed optimal solution for a given T is presented as a relatively small number of activities (chain-flows) which are repeated over and over until the end of the T periods; in particular, hold-overs at intermediate nodes are not required. The authors note that it is remarkable that there exist solutions to their problem that have this simple form, since other dynamic linear-programming problems that have been studied do not enjoy this property. *M. M. Flood* (Ann Arbor, Mich.)

1404:

Gastinel, Noël. Procédé itératif pour la résolution numérique d'un système d'équations linéaires. C. R. Acad. Sci. Paris **246** (1958), 2571–2574.

Verfasser skizziert ein Iterationsverfahren zur Bestimmung der Lösung x^* eines linearen Systems mit n Gleichungen und Unbekannten: $Ax-b=\sigma$, A eine reguläre $n\times n$ Matrix, x und b einspaltige Matrizen. Bezeichnet $f_i(x)$ die linke Seite der i-ten Gleichung, so ist $\phi(x)=\sum_1^n |f_i(x)|$ die absolute Fehlersumme. Für ein festes x des euklidischen Raumes E^n ist die Menge $\{x'|\phi(x')=\phi(x)\}$ eine konvexe Polyederfläche mit dem Symmetriezentrum x^*. Setzt man $\mu_i=1$ im Falle $f_i(x)\geqq 0$ und $\mu_i=-1$ im Falle $f_i(x)<0$, $i=1, \cdots, n$, und bezeichnet $\bar{x}$ die mit diesen μ_i gebildete Spalte, so steht der Vektor $A^t\bar{x}$ senkrecht auf einer der Polyederseiten, die x enthalten und

$$y=L(x)=x-\phi(x)\frac{A^t\bar{x}}{\|A^t\bar{x}\|^2}$$

ist die Orthogonalprojektion von x auf eine Hyperebene durch x^*, die jener Polyederfläche parallel ist. Die Punkte

$$x_1=L(x_0),\ x_2=L(x_1),\ \cdots,\ x_{j+1}=L(x_j),\ \cdots$$

bilden eine Folge, die gegen x^* konvergiert und es ist

$$\|x^*-x_j\|\leqq\|x^*-x_0\|\left(1-\frac{\lambda}{n\Lambda}\right)^{n/2}$$

wenn λ und Λ den kleinsten und grössten Eigenwert von A^tA bezeichnen.

Das Verfahren wird auf einer elektronischen Rechenmaschine getestet. *A. Pfluger* (Zürich)

1405:

Franckx, Ed. Résolution pratique des systèmes linéaires par la méthode des matrices de relaxation. Bull. Soc. Roy. Sci. Liège **26** (1957), 390–395.

The m by m matrix R is a relaxation matrix of A if $\sum_i |b_{ik}|<\Theta b_{kk}$, $\Theta<2$. To solve $Ax=c$ use a sequence R_1, R_2, $\cdots$ of relaxation matrices and set $x_0=0$, $x_{n+1}=x_n+\lambda_n r_k^i$ where r_k^i is the i-column of R_k, $|e_{nl}|=\max_j|e_{nj}|$, $e_n=Ax_n-c$, $e_{n+1,i}=0$. Then $|e_n|<|c|(1-(2-\Theta)/m)^n$, $|c|=\sum|c_j|$. A similar estimate holds for over- and under-relaxation. *T. S. Motzkin* (Los Angeles, Calif.)

1406:

Furuya, Shigeru. Methods of numerical calculation for simultaneous linear equations and inverse matrices. Sûgaku **9** (1957/58), 240–249. (Japanese)

Expository. *Rimhak Ree* (Vancouver, B.C.)

1407:

Pokorná, Olga. The solution of systems of linear algebraic equations. Stroje na Zpracování Informací **3** (1955), 139–196 (1956). (Czech. Russian and English summaries)

A review article, presenting a survey of direct and indirect methods of solution of systems of linear algebraic equations. The main direct methods are those of Gauss, orthogonalisation of Schmidt, A-orthogonal vectors, escalator of Morris and partition of the matrix. The indirect methods are those of iteration of Ritz, Seidel, Gauss-Seidel, Frazer-Duncan-Collaro, relaxation of Southwell, minimisation of the sum of squares, steepest descents, perpendiculars of Kaczmarz, Cimmin, Hotelling, general iteration of Cesari. Many methods given in the literature as different are actually the same. Illustrative examples of the solution of a system of four unknowns are given, using all of the methods discussed. There are 16 references up to 1953. *V. Vand* (University Park, Pa.)

1408:

Todd, John. The condition of certain matrices. III. J. Res. Nat. Bur. Standards **60** (1958), 1–7.

[Frühere Teile: Quart. J. Mech. Appl. Math. **2** (1949), 469–472; Arch. Math. **5** (1954), 249–257; MR **11**, 619; **16**, 523.] Bedeutet λ das Maximum der Eigenwertbeträge einer Matrix M und μ das Minimum, so ist die Zahl $P=\lambda/\mu$ ein Mass für die Schwierigkeit der numerischen Inversion von M, ein Mass für den Fehler der mit einer Eliminationsmethode berechneten Inversen. Diese Zahl P wird für einige Matrizen berechnet, welche bei Anwendung verschiedener Differenzenverfahren auf die Randwertaufgabe "$z_{xx}+z_{yy}=0$ im Einheitsquadrat, $z=f$ auf dessen Rand" auftreten. Z.B. ergibt sich bei der Maschenweite $h=1/(n+1)$ für das gewöhnliche Differenzenverfahren $P=4\pi^{-2}n^2$, für das gebräuchlichste Mehrstellenverfahren $P=8\pi^{-2}n^2/3$. *J. Schröder* (Hamburg)

1409:

***Bauer, F. L. Zusammenhänge zwischen einigen Iterationsverfahren der linearen Algebra. Aktuelle Probleme der Rechentechnik.** Bericht über das Internationale Mathematiker-Kolloquium, Dresden, 22. bis 27. November 1955, pp. 99–111. VEB Deutscher Verlag der Wissenschaften, Berlin, 1957.

The author investigates an iteration process generating a sequence of lower triangular matrices Δ_i. The basic rule is the following. Let A be a given matrix of general type. Compute the product $A\Delta_i$ and split this product into a lower triangle Δ_{i+1} and an upper triangle R_{i+1}, using the well-known algorithm of Gauss-Banachiewicz. If the sequence Δ_i converges to a triangle Δ, one has, finally,

$$A\Delta=\Delta R;$$

this is to say that the diagonal elements of R are the eigenvalues of the given matrix A. Hence, the algorithm may be used in order to compute eigenvalues. The author investigates the convergence of the iteration and shows that, by a little more sophisticated modification, it is possible to obtain quadratic convergence. He shows that Rutishauser's "LR-transformation" [C. R. Acad. Sci. Paris **240** (1955), 34–36; Z. Angew. Math. Phys. **5** (1954), 233–251, 496–508; **6** (1955), 387–401; **7** (1956), 164–169; MR **16**, 785, 176, 863; **17**, 789; **18**, 418]is a special case of his more general iteration. Furthermore several classical methods of numerical analysis (those of Bernoulli and Graeffe for solving algebraic equations, QD-algorithm, method of conjugate gradients, Richardson and Gauss-Seidel iteration of linear equations) turn out to be connected with this Δ_i algorithm. We are very grateful to the author for his effort to find general viewpoints in order to give us a better understanding of the tremendous variety of numerical methods in linear algebra.

E. Stiefel (Zürich)

1410:

Henrici, Peter. The quotient-difference algorithm. Nat. Bur. Standards Appl. Math. Ser. no. **49** (1958), 23–46.

The author describes the QD-Algorithm of Rutishauser [Z. Angew. Math. Phys. **5** (1954), 233–251, 496–508; MR **16**, 176, 863]. Unlike Rutishauser, the present author starts from a well-known property of Hankel determinants and deduces all properties of the QD-Algorithm in a simple and natural way. — By introduction of the polynomials

$$H_k^{(n)}(\lambda)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+k-1} & 1 \\ a_{n+1} & a_{n+2} & \cdots & a_{n+k} & \lambda \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_{n+k} & a_{n+k+1} & \cdots & a_{n+2k-1} & \lambda^k \end{vmatrix},$$

for which an important new theorem is proved, the author also derives the properties of Rutishauser's polynomials $p_\sigma^{(\nu)}(z)$. — In an appendix, the author refers to a continuous analog of the QD-Algorithm [Arch. Math. **5** (1954), 132–137; MR **16**, 176] as well as to a new interpretation in terms of matrices which leads to the LR-Transformation.

E. Stiefel (Zürich)

1411:

Korvasová, Květa; and Svoboda, Antonín. Determining the complex roots of algebraic equations on punched-card computing machines. Stroje na Zpracování Informací **3** (1955), 129–138 (1956). (Czech. Russian and English summaries)

A method of approximate evaluation of real and complex roots of algebraic equations of nth order with real coefficients is discussed. It consists of computing on punched-card computing machines the values of the equation at all points of a fixed network, in the Gauss complex plane. A polar grid is suitable for this purpose. This is followed by interpolation for zeros of the function. The method is advantageous for $n\leq 10$.

V. Vand (University Park, Pa.)

1412:

Larionov, B. A. On a method of approximate solution of an algebraic equation. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **21** (1957), 71–74. (Russian)

Let $f(x)=\sum_{k=0}^{n} q_k x^{n-k}$ and set

$$(1+v)^n f\left(\frac{a+bv}{1+v}\right)=g(v)=\sum_{k=0}^{n} Q_k v^{n-k} \quad (a<b).$$

The author discusses the relationship between the roots of $f(x)=0$ and $g(v)=0$. In particular set $a=0$, and let b be an upper bound for the positive roots of $f(x)=0$. Suppose that v_1 is an upper bound for the positive roots of the transformed equation $g(v)=0$. Then we have an improved upper bound $b_1=bv_1/(1+v_1)$ for the positive roots of $f(x)=0$. By repeating this process the author obtains a decreasing sequence which converges to the largest positive root of $f(x)=0$.

A. W. Goodman (Lexington, Ky.)

1413:

Plainevaux, Jean E. Intégration graphique de l'inverse d'une fonction. C. R. Acad. Sci. Paris **246** (1958), 2210–2212.

A numerical integration procedure is established for the integral

$$y=\int \frac{dx}{g(x)}$$

proceeding on stated increments in y to determine corresponding values of x. The method is given a graphical interpretation and methods of increasing accuracy are discussed.

P. C. Hammer (Madison, Wis.)

1414:

Dikiĭ, L. A. A new method for an approximate computation of eigenvalues in the Sturm-Liouville problem. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 12–14. (Russian)

The author considers the problem (P) $-u''+p(x)u=\lambda u$, $u(0)=u(\pi)=0$, where p is assumed to have the form $p(x)=\sum_{n=1}^{N} a_n \cos nx$. An approximation method for computing the eigenvalues $\{\lambda_n\}$ of (P) is given. It is based on the asymptotic expansion $\lambda_n\sim n^2+(c_2/n^2)+(c_4/n^4)+\cdots$, where $c_2=(4\pi)^{-1}\int_0^\pi p^2(x)dx$, etc., and on formulas obtained previously by the author [Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 187–200; MR **17**, 370], such as $\sum_{n=1}^{\infty}(\lambda_n-n^2)=\frac{1}{4}[p(0)+p(\pi)]$, and similar expressions involving higher powers of the eigenvalues. The method is illustrated by using the third approximation to compute the eigenvalues λ_1, λ_2, λ_3 of the problem for which $p(x)=\cos 2x$ (Mathieu). This involves solving a cubic equation, and the accuracy obtained here is to four figures.

E. A. Coddington (Princeton, N.J.)

1415:

Box, G. E. P.; and Coutie, G. A. Application of digital computers in the exploration of functional relationships. Proc. Inst. Elec. Engrs. B. **103** (1957), supplement no. 1, 100–107.

The authors describe an experimental and numerical

procedure whereby parameters $\theta_1, \cdots, \theta_p$ may be estimated, determining the exact differential equations

$$\frac{d\eta_u}{dt}=\varphi_u(\eta_1, \cdots, \eta_s; \xi_1, \cdots, \xi_k; \theta_1, \cdots, \theta_p) \quad (u=1, 2, \cdots, s),$$

which govern a process in which the functions φ_u are known and the η_u can be measured. For a given set of initial conditions, initial estimates of θ (appropriately scaled), and an appropriate set of measurements y of the η_u, the method is one of least squares and steepest descent applied to the expression $S(\theta)=\sum[y-f(\xi, \theta)]^2$, where $\eta=f(\xi, \theta)$ is the (vector) solution of the system of differential equations. Specifically, beginning with an estimated θ and appropriate nearby values, one solves the equations numerically, uses least squares to estimate the coefficients of an approximation to the resulting tabulated f as a truncated power series in θ, and applies a steepest descent method to the corresponding estimated formulae for S as a function of θ_u in order to obtain the next estimates of θ_u. This procedure is repeated. The accuracy of the results is controlled by increasing the degree of the approximation to f and by using statistical tests for goodness of fit and confidence intervals.

The method has been tested by the authors on a system of equations for chemical reactions. The computations employed subroutines for the Runge-Kutta-Gill procedure, to solve the differential equations, and a linear-equation-solving procedure, to solve the normal equations as soon as least squares approximations to f became non-linear. The subroutines were run on the "Deuce" machine at the National Physical Laboratory in England.

S. Gorn (Philadelphia, Pa.)

1416:

Četaev, N. G. On estimates of approximate integrations. Prikl. Mat. Meh. **21** (1957), 419–421. (Russian)

The author presents an error growth theorem for the integration of a system of ordinary differential equations

$$(1) \qquad \frac{dx_s}{dt}=X_s(t, x_1, \cdots, x_n) \quad (s=1, \cdots, n)$$

when the X_s are holomorphic functions of the real variables x_s in some region D for all time t. If an approximate solution

$$u_s(t)=x_s(t)-\xi_s(t)$$

is obtained by any approximating technique, he shows that for certain conditions of stability on the linearized error equations corresponding to (1), certain initial errors $\xi_s(0)$, and a certain bounded approximation error (due to either truncation or round-off) locally, the total error $\sum\xi_s^2$ at any later stage remains a bounded function of the latter two errors. This would indicate convergence of such an approximation process upon decreasing any or all of the above errors. This result appears to be a generalization of a number of theorems in approximation of solutions of differential equations by difference equations (for example, the reviewer's [≠1417, below], with a slightly different norm.) The proof is incomplete, referring to work of Liapunov and a work on stability of the author's, but no bibliography appears. Two examples are given, approximation to an nth order differential equation by ignoring the (small) coefficient of the highest derivative, and approximation of a first-order equation by power series. Application of this result to numerical approximations might prove very useful.

J. W. Carr, III (Ann Arbor, Mich.)

1417:

Carr, John W., III. Error bounds for the Runge-Kutta single-step integration process. J. Assoc. Comput. Mach. **5** (1958), 39–44.

Consider $dy/dx=f(x, y)$, and assume that $\partial f/\partial y$ exists and is continuous in a region D. The paper establishes the "stability" of the fourth order Runge-Kutta method of stepwise integration of the differential equation, in the following sense. Let y_i denote the exact solution of the differential equation at $x_i=x_0+ih$, and let y_i^* denote the calculated solution at x_i of the difference equations characteristic of the Runge-Kutta fourth-order procedure. Then the following theorem holds: If $-M_2<\partial f/\partial y<-M_1<0$, where $M_2>M_1>0$, if the stepsize h is taken as $h<\min(M_1/M_2^2, 4M_1^3/M_2^4)$, and if E denotes the maximum absolute error at each step (due to truncation and rounding), then $|y_i-y_i^*|\leq 2E/hM_1$. The stability of procedures employing difference equations was previously also treated by the reviewer [Proc. Amer. Math. Soc. **5** (1954), 869–887; MR **16**, 630]. *M. Lotkin* (Stratford, Conn.)

1418:

Warner, F. J. On the solution of "jury" problems with many degrees of freedom. Math. Tables Aids Comput. **11** (1957), 268–271.

The use of step-by-step methods for solving non-linear boundary-value problems in ordinary differential equations requires the determination of one or more parameters, which are chosen so that the boundary conditions are satisfied. The choice involves trial runs followed by a process of inverse interpolation. Particular applications of the latter are considered, with special reference, illustrated by examples, to the case of several parameters involving an extension of Aitken's method for the one-parameter case. *L. Fox* (Teddington)

1419:

Glinskaya, N. N.; and Mysovskih, I. P. On numerical solution of a boundary problem for a non-linear ordinary differential equation. Vestnik Leningrad. Univ. Ser. Mat. Fiz. Him. **9** (1954), no. 8, 49–54. (Russian)

1420:

Vitásek, Emil. Einfluss der Formulierung der Randbedingungen auf die Konvergenzgeschwindigkeit bei der Lösung von partiellen Differentialgleichungen mittels der Differenzenmethode. Apl. Mat. **2** (1957), 163–183. (Czech. Russian and German summaries)

Die vorliegende Untersuchung zerfällt in fünf Abschnitte. Zuerst wird die Lösung der partiellen Differentialgleichung (*) $\partial^2u/\partial x^2=\partial u/\partial t$ im Bereich R: $(0<x<a, 0<t<T)$ unter den Anfangsbedingungen

$$u(x, 0)=p(x) \text{ für } 0<x<a;$$

$$\frac{\partial u(0, t)}{\partial n}=Q_1(t), \quad \frac{\partial u(a, t)}{\partial n}=Q_2(t), \quad 0<t<T,$$

behandelt. Wird p gewählt, und die Ebene (x, t) innerhalb R mit Geraden $x=mh$, $t=n\tau$ überdeckt, wobei $h=a/p$, $\tau=\beta h^2$, $0<\beta<\frac{1}{2}$, m, n ganz, so bestimmen die Schnittpunkte dieser Geraden ein von den Parametern h und τ abhängiges Netzmaschensystem. Die Methode der Lösung der Differentialgleichung (*) mit Hilfe einer solchen Netzüberdeckung (Differenzenmethode) kommt auf die Kenntnis einer Funktion $U(x, t)$ hinaus, die in den Maschenpunkten innerhalb $\bar{R}$: $(0\leq x\leq a, 0\leq t\leq T)$ definiert ist

und in jedem in R gelegenen Maschenpunkt den Bedingungen

$$\frac{U(x+h,t)-2U(x,t)+U(x-h,t)}{h^2}=\frac{U(x,t+\tau)-U(x,t)}{\tau}$$

und $U(x,0)=p(x)$ auf der Geraden $t=0$, $0\leqq x\leqq a$ genügt. An Stelle der Anfangsbedingungen treten die Differenzengleichungen

$$\frac{U(h,t)-U(0,t)}{h}=-Q_1(t),\quad \frac{U(a,t)-U(a-h,t)}{h}=Q_2(t).$$

Mit U und u ist auch $\varepsilon(x,t)=U(x,t)-u(x,t)$ in jedem Maschenpunkt innerhalb R definiert. Für $\varepsilon(x,t)$ wird bewiesen: die Funktion $\varepsilon(x,t)$ genügt in R der Abschätzung $|\varepsilon(x,t)|\leqq Mh$, wobei M nur von R, von u und den Ableitungen dieser Funktion bis zu vierten Ordnung, aber für hinreichend kleine h nicht von h abhängt. Die Funktion $u(x,t)$ kann dabei den angegebenen Bedingungen unterworfen werden, wobei für $t\geqq t_0>0$ die Geschwindigkeit der Nullkonvergenz von $\varepsilon(x,t)$ genau durch h gegeben ist. Als Anwendungen der Theorie bieten sich die zweiten und dritten Randwertaufgaben der Theorie der Wärmeleitung im ein- und zweidimensionalen Fall sowie das dritte Randwertproblem für die Laplacesche Gleichung $\partial^2u/\partial x^2+\partial^2u/\partial y^2=0$. Zum Schluß wird ein numerisches Beispiel für den Fall der Differentialgleichung (*) mit der Anfangsbedingung $u(x,0)=e^x$ und den Randbedingungen $\partial u(0,t)/\partial x=e^t$, $\partial u(1,t)/\partial x=e^{t+1}$ im Intervall $\langle 0,1\rangle\times\langle 0;0{,}24\rangle$ tabellarisch durchgerechnet. Die Arbeit ist für Verfasser durch die Unzweckmäßigkeit des Ersatzes der Ableitungen in der Differenzenmethode durch die zugehörigen Differenzen bei der Formulierung der Randbedingungen veranlaßt worden. Dabei wurden nämlich bisher der Differentialoperator mit der Genauigkeit h^2, dagegen die ersten Ableitungen in den Randbedingungen mit der Genauigkeit h approximiert. Auch die Interpolation der ersten Randbedingungen durch eine Parabel im Falle der Wärmeleitunggleichung ist unzweckmäßig. *M. Pinl* (Köln)

1421:

Maruašvili, T. I. On the accuracy of a finite-difference equation which approximates the Poisson equation on electric grids. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **24** (1957), 163–177. (Georgian. Russian summary)

1422:

Ohocimskiĭ, D. E.; Kondraševa, I. L.; Vlasova, Z. P.; and Kazakova, R. K. Computation of point explosion taking into account counter-pressure. Trudy Mat. Inst. Steklov. no. **50**. Izdat. Akad. Nauk SSSR, Moscow, 1957. 66 pp. 3.20 rubles. (Russian)

The equations for spherically symmetrical flow are expressed in Lagrangian form for the following dependent variables: two Riemann invariants, pressure, and Eulerian radius. These equations, the shock conditions, and the boundary conditions at the center are converted into implicit difference equations relating conditions at time $\tau+\Delta\tau$ to those at τ. Simplified general forms of the solution are developed which reduce the most complicated part of the calculation at each forward time step to solving iteratively a set of four algebraic equations. A familiar exact explicit "automodel" solution with no counter pressure is used to provide initial data for very small τ_0 with intense shock. To keep the number of space intervals constant, at each forward time step one pair of adjacent time intervals was merged. Time and space intervals were systematically increased as the work progressed. Accuracy checks were based on energy balance, comparison of Eulerian and Lagrangian radius at the shock, and smoothness of the particle velocity near the edge of the core of the explosion. Calculated results for a particular initial energy, presented graphically and tabularly, are said to agree well with those computed by H. H. Goldstine and J. von Neumann [Comm. Pure Appl. Math. **8** (1955), 327–353; MR **17**, 314] with a much finer net and about eight times as many time steps.

J. H. Giese (Aberdeen, Md.)

1423:

Thompson, Gene Thomas. On Bateman's method for solving linear integral equations. J. Assoc. Comput. Mach. **4** (1957), 314–328.

Bateman's method [H. Bateman, Proc. Roy. Soc. London, Ser. A **100** (1922), 441–449] for the numerical solution of the integral equation

$$x(s)-\int_0^1 W(s,t)x(t)dt=y(s)$$

consists in replacing the kernel $W(s,t)$ by the kernel of finite rank

$$H_n(s,t)=\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\alpha_{ij}}{W_n}W(s,t_j)W(s_i,t),$$

where $W_n=\det[W(s_i,t_j)]$, and α_{ij} is the cofactor of $W(s_i,t_j)$ in W_n; this has the properties

$$H_n(s,t_j)=W(s,t_j),\quad H_n(s_i,t)=W(s_i,t).$$

The equation

$$x_n(s)-\int_0^1 H_n(s,t)x_n(t)=y(s)$$

is then solved, and $x_n(s)$ is taken as an approximation to the solution $x(s)$ of the exact equation.

The author obtains bounds for the mean square error $\|x_n-x\|$ in the general case, and improved bounds in the special case when $W(s,t)$ is the Green's function of a Sturm-Liouville differential equation. A numerical illustration is given. *F. Smithies* (Cambridge, England)

1424:

Fabian, Václav. Zufälliges Abrunden und die Konvergenz des linearen (Seidelschen) Iterationsverfahrens. Math. Nachr. **16** (1957), 265–270.

For the total step iteration, a stochastic process for rounding the iterates was studied by Abramov [Vyčisl. Mat. Vyčisl. Tehn. **1** (1953), 37–40; MR **16**, 1156]. This author takes the probability of rounding the last retained digit up or down to be proportional to the truncated portion of the number, and shows that convergence in the mean to the true solution is almost certain. The form of the iteration is quite general: he assumes a sequence of operators $A_1, A_2, \cdots, A_s$ cyclically applied, with the understanding (necessary and sufficient for securing convergence of the exact, unrounded, iteration) that

$$(A_sA_{s-1}\cdots A_1)^k\to 0 \text{ as } k\to\infty.$$

A. S. Householder (Oak Ridge, Tenn.)

1425a:

Nikolaev, P. V. On the projectivity of binary anamorphoses of functions. Ural. Politehn. Inst. Trudy **51** (1954), 51–64. (Russian)

1425b:
Nikolaev, P. V. Binary anamorphoses of functions. Ural. Politehn. Inst. Trudy **51** (1954), 65–76. (Russian)
By a binary anamorphosis of a function

$$F(t, \tau)=F(t_1, \tau_1, \cdots, t_n, \tau_n),\ n\geqq 3,$$

is meant a transformation of F into a determinant

$$F(t, \tau)=|f_{t1}(t_t, \tau_t), \cdots, f_{tn}(t_t, \tau_t)|.$$

These articles are an extension of the author's earlier results on existence and effective calculation of anamorphoses [see Dokl. Akad. Nauk SSSR (N.S.) **68** (1949), 229–232; MR **11**, 406].

1426:
Smirnov, S. V. On the existence of a solution to the problem of general anamorphosis. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 359–362. (Russian)

1427:
Lohmander, Bengt; and Rittsten, Stig. Table of the function $y=e^{-x^3}\int_0^x e^{t^3}dt$. Kungl. Fysiogr. Sällsk. i Lund Förh. **28** (1958), 45–52.
This tabulates $e^{-x^3}\int_0^x e^{t^3}dt$ to 10 decimals, with $\delta_m{}^2$, for $x=0(.01)5$ and for $1/x=0(.005).2$. There are also 20-decimal values of y for $x=0(0.5)10$, and of x, y at the maximum, and of x, y, and xy at the inflexion.
J. C. P. Miller (Cambridge, England)

1428:
***Integrals of Airy functions.** National Bureau of Standards, Applied Mathematics Series no. 52. U. S. Government Printing Office, Washington, D. C., 1958, iii+28 pp. $0.25.
This book has two parts.
If we define $\mathrm{Ai}(x)=\pi^{-1}\int_0^\infty \cos(\frac{1}{3}t^3+xt)dt$, which is a solution of $y''=xy$, then the first part gives 7 decimal tables with δ^2 of the functions $f(x)=\int_0^x \mathrm{Ai}(-t)dt$ and $F(x)=\int_0^x f(t)dt$. This part has a short introduction by E. E. Osborne.
The second part, with an Introduction by P. Rabinowitz is concerned with $A_0(x)=\int_0^x \exp(-\frac{1}{3}t^3-xt)dt$ which satisfies the differential equation $y''+xy=1$. The tables give $A_0(x)$ and $A_0'(x)$ to 8 decimals with δ^2 for $x=0(.01)1(.02)5(.05)11$ and for $1/x=0(.01).1$, with $\delta_m{}^2$. There is also a small table of $\int_0^x A_0(t)dt$, to 8 decimals for $x=\frac{1}{2}$, 1(1)11.
J. C. P. Miller (Cambridge, England)

1429:
***Emersleben, Otto. Werte einer Zetafunktion 2. Ordnung mit Argument $s=2$, der Grundfunktion der doppelperiodischen Parallelströmung zäher Flüssigkeiten.** Anwendungen der Mathematik. Reihe: Funktionstabulierungen. Universität Greifswald, Greifswald, 1956. 13 pp. (1 insert)
This gives, on a single, multiply-folded, sheet, a table of the Epstein Zeta-function

$$Z\begin{vmatrix}0 & 0\\ x & y\end{vmatrix}(2)_\delta=\sum_{k=-\infty}^{\infty}{\sum_{l=-\infty}^{\infty}}'\frac{\exp\{2\pi i(kx+ly)\}}{k^2+l^2}$$

$k=l=0$ being excluded. Values are to 6 decimals generally, with 7 or 8 in a few places, for $y=0(.01).5$, $x=y(.01).5$. The values for $(x, y)=(\frac{1}{2}, \frac{1}{2})$, $(0, \frac{1}{2})$, $(\frac{1}{4}, \frac{1}{4})$ which are respectively $-\pi \ln 2$, $-\frac{1}{2}\pi \ln 2$ and $-\frac{1}{4}\pi \ln 2$ are given to 25 decimals.
The insert lists a number of errata.
J. C. P. Miller (Cambridge, England)

COMPUTING MACHINES

See also 832, 1411, 1415, 1602.

1430:
Černý, Václav. Machine for testing the central memory of the Czechoslovak automatic computer SAPO. Stroje na Zpracování Informací 3 (1955), 77–88 (1956). (Czech. Russian and English summaries)
A machine for testing the central magnetic memory drum of the Czechoslovak automatic digital computer SAPO is described. It imitates the operation of the relay part of the computer, so that the drum can be tested before the entire computer is completed. Block diagrams are given and the operation is described.
V. Vand (University Park, Pa.)

1431:
Sobol', I. M. Pseudo-random numbers for the machine "Strela". Teor. Veroyatnost. i Primenen. 3 (1958), 205–211. (Russian. English summary)
The Strela-1, the digital computing machine of the Mathematical Institute (Computing Center) of the Academy of Sciences of the USSR, lacks a fixed-point multiplication instruction. The author describes a special technique using a binary floating-point multiplication instruction, that generates a positive 35-bit (size of the floating-point mantissa of numbers) pseudo-random number γ_{k+1} by means of the following sequence of operations on a previous number γ_k: 1) Floating-point multiplication of γ_k by 10^{17}; 2) shifting the product (fixed-point) seven places left, to zero out the exponential part; 3) formation of the absolute value of the result, which includes, in many cases, normalization — extra left — shifts to place the first non-zero bit in the most significant position, yielding a value $0\leqq\gamma_{k+1}<1$.
On the basis of experiment, the author feels that this very non-linear algorithm is as effective as either von Neumann's mid-square method or Lehmer's congruence procedure, both of which are based on fixed-point arithmetic. He found aperiodic runs, dependent on γ_0, ranging up to 87, 834, and cycles of about 50,000. Upon conversion of the elements of the sequence to decimal fractions, χ^2 tests on 50,000 sequential pairs (γ_k, γ_{k+1}) distributed over a 10×10 grid, using the standard frequency, serial, gap, poker and run tests, as well as a special test based on directional bias, gave values useful for both large and smaller samples.
The resulting pseudo-random sequences were applied "more than once to solution of integral equations by the method of Monte Carlo" and "passage of particles through a flat layer." The initial value γ_0 was chosen at random from a table of random numbers.
J. W. Carr, III (Ann Arbor, Mich.)

1432:
***Thüring, Bruno. Einführung in die Methoden der Programmierung kaufmännischer und wissenschaftlicher Probleme für elektronische Rechenanlagen. Teil 1: Die Logik der Programmierung.** Robert Göller Verlag, Baden-Baden, 1957. 217 pp. DM 45.00.
This book is a thorough introduction to digital computer programming that provides a very complete review of the fundamentals of what has been, and still is, an art rather than a science. This book, like its predecessors, does not attempt any fundamental mathematical description of what is going on. Unlike most of its predecessors, it does divorce itself almost completely from a description of any real or "hypothetical" machine, adopting the "algorithmic" approach. The notation is very similar

to that of the "Zurich-type" compilers now under development in the U.S.A., Western Germany, and Switzerland, being similar to that originally credited to K. Zuse.

This detachment from any specific machine is a tremendous gain. The book is full of flow diagrams as well as written algorithms, and covers both scientific computations and "business problems," the latter being mainly combinational sorting problems. The description of programs in terms of logical relations is very useful, and the sorting discussion is very complete. If any computer is mentioned, reference is made briefly to the Univac I and II computers. Thus the book also describes magnetic tape usage in detail.

J. W. Carr, III (Ann Arbor, Mich.)

1433:

Mercer, Robert J. Micro-programming. J. Assoc. Comput. Mach. **4** (1957), 157–171.

The author reviews Wilkes' logical design concept known as micro-programming [M. V. Wilkes and J. B. Stringer, Proc. Cambridge, Philos. Soc. **49** (1953), 230–238; MR **14**, 691]. Defining components of machine operation M_i to be the "micro-orders" m_{ij}, each of which consists of the simultaneous execution of a number of "micro-operations", the author describes how a set of machine operations M_i can be determined by a sequencing matrix S, a connection matrix C to the arithmetic and control registers, and a T matrix which determines the variety of transpositions in transferring information between registers. A simple adder, in which carries are handled in parallel at each pulsetime, is described in micro-operations.

The second half of the paper describes in detail the S, C, T matrix and register connections of a machine with five operations built up out of thirteen micro-orders. This machine is designed to be an extremely efficient generator to all permutations on n digits. The flow chart and program of this permutation-generating micro-programmed machine are exhibited for the Paige-Tompkins generating scheme [C. B. Tompkins, Proc. Symposia Appl. Math. vol. 6, McGraw-Hill, New York, pp. 195–211; MR **18**, 238].

S. Gorn (Philadelphia, Pa.)

1434:

Dix, C. H. The numerical computation of Cagniard's integrals. Geophysics **23** (1958), 198–222.

The main interest in this paper lies in its varied uses of the "Electrolytic Tank" type analogue computor for problems in functions of a complex variable. To the best of this reviewer's knowledge the main idea of this analogue computor was suggested in a wartime report to the British Ministry of Supply by the late Professor Daniels (although, of course, much of Felix Klein's pioneer work in function theory in the 19th century used potential theory concepts); it was later developed and extended to complex function analysis of many forms by Boothroyd, the reviewer and Makar [Proc. Inst. Elec. Engrs. **96**, part I (1949), 163–177]. Briefly, any function of a complex variable, since it satisfies Laplace's equation, may be represented analogously by the current and potential distribution over a very thin conducting sheet (e.g. liquid electrolyte) if electrodes are inserted corresponding to poles, zeros and branch cuts. In principle, every property of a function, so represented, appears in physical analogous forms: saddle points, residues [Boothroyd, Cherry and Makar, loc. cit.], roots of polynomials [F. Lucas, C. R. Acad. Sci. Paris **106** (1888), 121–122, 195–197, 268–270, 587–589, 645–648, 1072–1074], conditions at infinity, branch points and cuts, conformal transformations— the whole apparatus of function theory. Further, an infinite plane may be simulated very simply, by using a circular thin electrolytic layer, divided into two separate layers by a circular insulating sheet raised just clear of the tank bottom; this achieves a precise physical boundary match [Boothroyd, Cherry and Makar, loc. cit.]

Such an analogue computer is simple and cheap; it can be developed to an accuracy of about 1% and there exist several manual computation techniques whereby accuracies may subsequently be improved. The method deserves to be more widely known and this paper may, apart from its immediate application and interest, serve to call the attention of mathematicians and computers to the techniques.

The paper concerns especially computations involved in the determination of surface wave patterns set up on the plane surface of a semi-infinite solid, when an impulsive compressional force occurs at a distance h below the surface — a problem of great interest to prospectors. The computations are carried out both by using the electrolytic tank computor and by graphical means. A very wide variety of computations are involved and it is to be hoped that other applications will suggest themselves to readers. It is refreshing to see this interest in analogue computation, for specialised uses, in a world which appears to have become so obsessed by digital computors!

Colin Cherry (London)

1435:

Vurcfeld, Václav. Analogue machine for the solution of algebraic equations of higher order by a weighing. Stroje na Zpracování Informací **3** (1955), 89–100 (1956). (Czech. Russian and English summaries)

An analogue calculating machine based on summation of torques by weighing is proposed. The machine can be used for finding real roots of higher order polynomials.

The polynomial is transformed into a Fourier series by a trigonometric substitution. Each term of the Fourier series is then represented by a balancing wheel, mounted on a common frame of a balance. Each wheel carries a weight, placed eccentrically and proportional to the Fourier coefficient, and is rotated, say, by a ratchet, proportionally to the frequency. The equilibrium positions of the balance indicate the roots of the Fourier series and thus the roots of the polynomial.

{Similar machines have already been constructed by J. Harvey [Proc. Phys. Soc. **42** (1930), 245–250] and by V. Vand [J. Sci. Instrum. **27** (1950), 257–261; MR **13**, 591]. There are no references to these in the paper.}

V. Vand (University Park, Pa.)

1436:

Genuys, François. Dix mille décimales de π. Chiffres **1** (1958), 17–22.

This paper gives π to 10000 decimals, computed from Machin's formula $\pi=16 \text{ arc cot } 5-4 \text{ arc cot } 239$ on an I.B.M. 704 computer. No indication of checking is given. but I have compared this value with that of G. E. Felton (from machine-script kindly supplied to me), which gives 10017 decimals, obtained twice — from Klingenstierna's and Gauss's formulae, namely

$$\pi=32 \text{ arc cot } 10-4 \text{ arc cot } 239-16 \text{ arc cot } 515$$
$$=48 \text{ arc cot } 18+32 \text{ arc cot } 57-20 \text{ arc cot } 239.$$

The agreement is complete to 10000 decimals.

A preliminary value of Felton's was published (before final checking) in "Oxford Mathematical Conference, abbreviated proceedings" [Trinity College, Oxford, 1957,

Times Publ. Co.]. The value is given as an extensive footnote (pages 14–53) and Felton himself found that only 7480 decimals are correct in this value.

J. C. P. Miller (Cambridge, England)

MECHANICS OF PARTICLES AND SYSTEMS

See also 1579, 1580, 1581, 1585, 1586, 1604, 1616.

1437:

Truesdell, C. Recent advances in rational mechanics. Science **127** (1958), 729–739.

A Sigma Xi initiation lecture delivered on 16 May 1956 at the State University of Iowa, Iowa City.

1438:

Goldberg, Michael. Rotors tangent to n fixed circles. J. Math. Phys. **37** (1958), 69–74.

The author considers the problem of constructing a curve (other than a circle) which can be continuously displaced while remaining tangent to n fixed circles. In the special case in which the circles are equal and their centers form a regular n-gon, this can be done by means of his earlier work on rotors in regular polygons. (In other cases the problem has not been solved.) For instance, one kind of rotor in a square is an oval having the symmetry of an equilateral triangle. Instead of keeping the square fixed and moving the oval, we may keep the oval fixed and move the square so that its four sides continue to touch the oval. Then the vertices of the square describe a larger oval, say ω. Now, keeping the square fixed, we can move the oval ω in such a way that it continues to pass through all the vertices of the square. Finally, replacing the square by four circles of radius a drawn round its vertices, and ω by two curves parallel to ω at distance a, we see that either of these parallel curves can be moved so as to remain tangent to the four circles. Incidentally, the curve ω is unusual in having an infinity of inscribed squares: according to Steinhaus [Mathematical snapshots, Oxford, 1950; MR **12**, 44; p. 87], a simple closed curve has, in general, a unique inscribed square.

H. S. M. Coxeter (Toronto, Ont.)

1439:

Mangeron, D. Über einige Eigenschaften der reduzierten Beschleunigungen beliebiger Ordnung. Bul. Inst. Politehn. Iaşi (N.S.) **3** (1957), no. 1–2, 43–46. (Romanian. Russian and German summaries)

Kurze Noten zur ebenen Kinematik. Der geometrische Ort der Endpunkte von reduzierten höheren Beschleunigungsvektoren der Punkte einer Gerade l ist eine Gerade l'. Betrachtungen über den Winkel zwischen l und l'.

O. *Bottema* (Delft)

1440:

Ciobanu, Gh. Sur la cinématique des mouvements de roulement et glissement des systèmes matériels. Bul. Inst. Politehn. Iaşi (N.S.) **2** (1956), no. 3–4, 39–41. (Romanian. Russian and French summaries)

1441:

Bradeanu, Petru. L'équation du pendule mathématique de masse variable. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. Fiz. **7** (1956), no. 1–4, 65–77. (Romanian. Russian and French summaries)

Dans le cas le plus simple il s'agit de l'équation $\ddot{\theta}+(1/m)\dot{m}\dot{\theta}+(g/l)\theta=0$; pour $m=m_0(1-\alpha t)$ l'intégration se fait par la fonction J_0 de Bessel. Généralisation: le pendule de masse variable dans un milieu résistant.

O. Bottema (Delft)

1442:

Braier, Alfred; et Antoniu, Mihai. Sur un aspect de la distribution des vitesses et des accélérations dans le mouvement d'un solide rigide. Bul. Inst. Politehn. Iaşi (N.S.) **2(6)** (1956), 43–48. (Romanian. Russian and French summaries)

1443:

Hemmer, P. Chr.; Maximon, L. C.; and Wergeland, H. Recurrence time of a dynamical system. Phys. Rev. (2) **111** (1958), 689–694.

The authors of this article supply the following summary. Poincaré cycles of a many-particle system are exemplified by the motion of a linear chain. It is shown that the recurrence time increases in an approximately exponential way with the number of degrees of freedom, and as a power of the sharpness with which the recurrent state is specified. An explicit formula is given which is applicable to other separable dynamical systems as well.

J. De Cicco (Chicago, Ill.)

1444:

Brearley, M. N.; and Bolt, B. A. The dynamics of a bowl. Quart. J. Mech. Appl. Math. **11** (1958), 351–363.

The "bowl" of the title is approximately what is known in the United States as a "bowling ball." However it is not spherical, but more nearly an oblate spheroid with a prolate spheroidal ellipsoid of inertia, having a small eccentricity. This is rolled along wheel-like. Because of the inclination of its axis to the horizontal and an assumed retarding couple, the bowl follows a curved rather than a straight path. The object is to determine the nature of this path.

The equations of motion are expressed in terms of Eulerian angles. They are complicated and nonlinear but the authors have succeeded in finding good approximations which are sufficiently simple. Furthermore they have conscientiously supplemented their theoretical work with an experimental program which shows excellent agreement.

E. Pinney (Berkeley, Calif.)

1445:

Faure, Robert. Synchronisation des systèmes mécaniques. Sur l'existence simultanée de deux types de solutions périodiques d'équations différentielles non linéaires à coefficients périodiques. C. R. Acad. Sci. Paris **245** (1957), 1293–1295.

The author considers mechanical systems, which, in the absence of coupling and forcing, are represented by a system, E, of linear homogeneous differential equations with real constant coefficients, the coordinate q_i satisfying an equation of order p_i, $D_i(q_i)=0$, $i=1, 2, \cdots, n$. The author studies periodic solutions of the mechanical system in the presence of weak coupling and forcing. The differential equation system is E': $D_i(q_i)=\lambda f_i(q_1, q_1', \cdots, q_n, \cdots, q_n^{(p_n-1)}, t)$, $i=1, 2, \cdots, n$. In E', λ is small and positive, each f_i is periodic in t of period T, and is analytic in t and all the q's and their derivatives of less than the maximal orders in the system E. The author proves that if E has no nontrivial solution of period T, then E' has a solution of period T which reduces to zero with λ. The proof is not given completely, but is by a Banach space technique introduced by the author in earlier notes [same C. R. **244** (1957), 2767–2769 and 3022–3025; MR **19**, 859].

The author next supposes that the system E has all its solutions periodic (hence every p_i is even), and that the periods of all such solutions are rational multiples of T. Using a method involving variation of constants, similar to the method of van der Pol, the author obtains under certain conditions the existence of solutions of E' which are stable and of period τ, where τ is the least common multiple of T and the periods of the solutions of E. These solutions do not tend to zero with λ. They have period T or are subharmonics with period a multiple of T.

W. S. Loud (Minneapolis, Minn.)

STATISTICAL THERMODYNAMICS AND MECHANICS

See also 1099, 1477.

1446:

Green, M. S. The non-equilibrium pair distribution function at low densities. Physica **24** (1958), 393–403.

The non-equilibrium states of a classical gas can be described by the hierarchy of n-particle distribution $f_n(1, 2, \cdots, n)$. Bogolyubov [Acad. Sci. USSR J. Phys. **10** (1946), 265–274; MR **9**, 72] has suggested that f_2, f_3, $\cdots$ can be written as functionals of f_1. In a previous article [J. Chem. Phys. **25** (1956), 836–855; MR **18**, 835] the present author has discussed this assumption for spatially uniform dilute gases. In this article the discussion is continued. Each collision gives rise to a correlation between the positions of the particles that have participated in it. It is shown how this correlation affects f_2 and f_3 at later times. The final conclusion is: "the non-equilibrium pair distribution function has correlations beyond the range of intermolecular forces; these diminish to zero at distances large compared to a mean free path but there are always traces of the correlations which existed at time zero." *N. G. van Kampen* (Utrecht)

1447:

Harper, P. G. Collision interference effects in a Bose-Einstein gas. Physica **24** (1958), 177–184.

The Boltzmann equation can be derived for a Bose-Einstein gas by considering binary collisions only and neglecting correlations between successive collisions. In order to estimate the influence of such correlations the author carries perturbation theory to fourth order, and finds that the non-Boltzmann interference terms are in general negligible. The reason is that the number of such terms is strongly restricted by conservation of momentum and energy, as in Van Hove's treatment [Physica **21** (1955), 517–540; MR **17**, 115]. However, in case of Bose condensation the non-Boltzmann terms may become important because of the large number of particles in the ground state. It is concluded that the Boltzmann equation does not correctly describe the approach to equilibrium below the condensation temperature.

N. G. van Kampen (Utrecht)

1448:

Bocchieri, P.; and Loinger, A. Ergodic theorem in quantum mechanics. Phys. Rev. (2) **111** (1958), 668–670.

It is shown that the quantum mechanical ergodic theorem follows solely from averaging over macro-observers, which means that there is no physical ground for the validity of the ergodic theorem in quantum mechanics, as there is no physical justification for equal a priori probabilities for all macro-observers.

D. ter Haar (Oxford)

1449:

Van Hove, Léon. A remark on the time-dependent pair distribution. Physica **24** (1958), 404–408.

This is a sequel to the author's earlier paper [Phys. Rev. (2) **95** (1954), 249–262; MR **16**, 206] on the connection between the scattering cross-section and the time dependent pair distribution function, particularly with reference to neutron scattering. Here it is shown that the imaginary part of the pair distribution function describes the change in density of the medium produced by the presence of a neutron. *G. Newell* (Stockholm)

1450:

MacDonald, D. K. C. Brownian movement. Phys. Rev. (2) **108** (1957), 541–545.

The paper attempts a general analysis of the Brownian motion for a system having a non-linear relaxation mechanism. Treatment is developed by means of a model, i.e., an electrical circuit with ideal capacitance and non-linear conductance. Detailed results are obtained for systems with a "modest" degree of non-linearity, i.e., such where the 2nd order terms for the non-linear element is small as compared with the 1st order term.

B. Gross (Rio de Janeiro)

1451:

Ziegler, Hans. An attempt to generalize Onsager's principle, and its significance for rheological problems. Z. Angew. Math. Phys. **9b** (1958), 748–763.

The paper discusses the problem of rheology in terms of irreversible thermodynamics. For linear problems, Onsager's relations apply; they are based on the assumptions that a) microscopic fluctuations about a state of equilibrium are reversible, and b) the regression of fluctuations obeys the same law as a macroscopic process. Onsager's relations can be generalized for non-linear phenomena, if the following analogous assumptions are introduced: a) The system is conservative and non-gyroscopic. b) The fluctuations are slow compared with the microscopic velocities. The significance of the generalized relations and their equivalence with Onsager's relations in an important particular case are discussed.

B. Gross (Rio de Janeiro)

1452:

Tisza, Laszlo; and Manning, Irwin. Fluctuations and irreversible thermodynamics. Phys. Rev. (2) **105** (1957), 1695–1705.

The authors state that "the basic problem of time dependent theory of fluctuations is to combine the phenomenological kinetic theory and the stochastic theory of random processes into a consistent scheme." It is indeed true that, as long as the phenomenological equations are linear, one can guess in a reasonably unique way a stochastic process which describes the temporal behavior of the fluctuations, as was done by Langevin [C. R. Acad. Sci. Paris **146** (1908), 530–533]. Alternative but mathematically equivalent descriptions of this stochastic process are the Fokker-Planck equation, and the path-integral approach popularized by Onsager and Machlup [Phys. Rev. (2) **91** (1953), 1505–1512; MR **15**, 273]. The latter is chosen by the present authors as the proper language for doing the guessing. If α is a macroscopic observable, it is postulated that $\alpha(t)$ constitutes a stochastic process, which is Markoffian and stationary, and obeys microscopic reversibility (=detailed balancing).

The space of functions $\alpha(t)$ is restricted to "exponentially smoothed paths"; which are the functions consisting of a number of successive pieces of the form $Ae^{-\gamma t}+Be^{\gamma t}$, where $1/\gamma$ is the phenomenological relaxation time. In this restricted function space the Onsager-Machlup probability distribution $\exp\{-\int[\dot{\alpha}(t)+\gamma\alpha(t)]^2dt\}$ is postulated. The practical results are not affected by the restriction, because the Onsager-Machlup distribution has sharp maxima for these exponentially smoothed paths. The method applies only to linear processes, but it is claimed to be simpler and better suited than the Langevin equation for describing how fluctuations can build up in time. The characteristic function and the spectral representation of the probability distribution are also given, and the relations of Einstein, Nyquist and Callen and Greene are derived. {The simplification in the latter case is not due to this particular model, but to the fact that Callen and Greene used an explicit model for deriving a general statistical result.}

N. G. van Kampen (Utrecht)

ELASTICITY, PLASTICITY

See also 1093, 1509.

1453:

***Freudenthal, Alfred M. Fatigue.** Handbuch der Physik, herausgegeben von S. Flügge. Bd. 6. Elastizität und Plastizität, pp. 591–613. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. DM 145.00.

The main topics are (i) micromechanics of progressive damage, with reference mainly to the concepts developed by Wood, Forsyth and Brown; (ii) statistical theories of fatigue life; (iii) cumulative damage under cycles of varying amplitude. The article is valuable as a guide to the literature rather than as a source of information in itself.

R. Hill (Nottingham)

1454:

***Irwin, George R. Fracture.** Handbuch der Physik, herausgegeben von S. Flügge. Bd. 6. Elastizität und Plastizität, pp. 551–590. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. DM 145.00.

Fracture is discussed here mainly from a macroscopic standpoint, and only briefly in terms of dislocation theory. The main subjects are (i) tensile strength of liquids; (ii) elastic stress fields round cracks (solutions of Griffith, Neuber, Sneddon and Westergaard [see, e.g., Sneddon, Proc. Roy. Soc. London Ser. A. **187** (1946), 229–260; MR **8**, 117]); (iii) crack formation and growth (qualitative discussion based on rather scanty experimental data); (iv) fracture criteria (in broad terms) ;(v) running cracks (photoelastic investigation); (vi) size effects (statistical analysis). The article gives a very good overall picture within the strict limits of present knowledge.

R. Hill (Nottingham)

1455:

Vvedenskaya, A. V. On dislocations on a surface of fracture accompanied by slipping. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1958**, 175–183. (Russian)

The writer shows that the field of dislocations determined in one of her previous papers [same Izv. **1956**, 277–284] with the use of the theory of dislocations actually describes the fractures in elastic media and the instantaneous translocations of both sides of the cut along the limited surface of the fracture. The fracture is accompanied by vibrations which originate on the perimeter of the fracture. If inside the limited surface of the fracture there is a uniform distribution of the couple of forces and moments, then at the beginning of the disturbances there takes place under the action of these forces an additional rotation of the surfaces of sliding with respect to the axis along which there is orientated the moment of the couple.

M. Z. v. Krzywoblocki (Urbana, Ill.)

1456:

Banditch, Ivan. Sur une équation différentielle indéterminée dans la théorie de l'élasticité. C. R. Acad. Sci. Paris **244** (1957), 2130–2133.

La solution de l'équation indéterminée de Truesdell

$$\frac{F''}{F}-\left(\frac{f'''}{f''}+\frac{f'}{f}\right)\frac{F'}{F}+n^2\frac{f''}{f}=0$$

($F=F(x)$, $f=f(x)$, $n=$const) est donnée par

$$f(x)=C_1w\theta,\quad F(x)=C_2w^2\theta^{1-n^2}\theta',$$

avec

$$\theta=\exp\int\left(\frac{1}{1-n^2}\frac{w''}{w}\right)^{\frac{1}{2}}dx$$

(w fonction arbitraire de x).

La solution de Mitrinovitch [C. R. Acad. Sci. Paris **232** (1951), 681–683; cf. also Bull. Soc. Math. Phys. Macédoine **2** (1951), 17–24; MR **12**, 558; **13**, 238] s'obtient par un procédé qui est plus compliqué que celui de Banditch.

D. S. Mitrinovitch (Belgrade)

1457:

Wegner, Udo. Theorie des Fliessspans. Z. Angew. Math. Mech. **38** (1958), 200–206.

Complex potentials and Mellin transforms are used to solve the elastic plane strain problem of an infinite sector of angle α, $\pi\leqq\alpha<2\pi$, under a force acting obliquely at a point on one side. No reference is made to the known solution of the generalized plane stress analogue [W. M. Shepherd, Proc. Roy. Soc. London Ser. A **148** (1935), 284–303]. Author believes analysis to be relevant to the mechanics of machining. The contour of maximum shearing stress through the vertex is especially examined for its supposed connexion with the localized region of plastic flow in the actual steady-state cutting process.

R. Hill (Nottingham)

1458:

Babuška, Ivo. Der Spannungszustand eines nichthomogenen Baugrundes. Apl. Mat. **2** (1957), 361–369. (Czech. Russian and German summaries)

In dieser Arbeit werden die Formeln, für den Spannungszustand einer nichthomogenen Halbebene mit dem Elastizitätsmodul $E(h)=E_0h^\alpha$ und der unveränderlichen Poissonschen Konstanten σ, angegeben.

Zusammenfassung des Autors

1459:

Uflyand, Ya. S. Axially-symmetric problem of elasticity theory for half-space with circular division line separating boundary conditions. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 531–533. (Russian)

The problem considered is that of the elastic equilibrium in x, y, z space of the half space $z\geqq0$, when the surface displacements are prescribed on $z=0$ within the circle $r=a$ and the surface tractions on $z=0$ are prescribed outside this circle. This mixed boundary value problem is attacked using the Papkovich-Neuber [H. Neuber, Concentrated loads, Gostehizdat, Moscow, 1947] representation of the displacement vector in terms of

harmonic functions; toroidal coordinates; and the Mehler-Fok integral expansion in terms of Legendre functions [N. N. Lebedev, Special functions and their applications, Gostehizdat, Moscow, 1953; MR **16**, 355]. The circular punch problem is dealt with as a special case.

J. B. Diaz (College Park, Md.)

1460:

Allen, D. N. de G.; and Dennis, S. C. R. The application of relaxation methods to the solution of differential equations in three dimensions. III. Three-dimensional stress analysis. Quart. J. Mech. Appl. Math. **11** (1958), 172–184.

[For parts I and II, see same J. **4** (1951), 199–208; **6** (1953), 81–100; MR **13**, 165; **14**, 1021]. The method of relaxation is applied to the determination of stresses in three dimensions. Introducing the stress functions of Maxwell, (χ, ϕ, ψ), it is shown that these satisfy the equations $\nabla^2\chi=\nabla^2\phi=\nabla^2\psi=\Theta/(1+\sigma)$, and $\nabla^2\Theta=0$, where σ is Poisson's ratio.

The boundary conditions for a cube are examined and the solution given for a cube with equal and opposite forces applied at the centres of two parallel faces.

D. C. Gilles (Glasgow)

1461:

Angelitch, T. P. The Beltrami-Michell compatibility equations in general tensor form obtained from Saint-Venant's compatibility equations. Srpska Akad. Nauka. Zb. Rad. **55**, Mat. Inst. **6** (1957), 1–4. (Serbo-Croatian. English summary)

In previous papers [Acad. Serbe Sci. Publ. Inst. Math. **5** (1953), 1–4; **9** (1956), 93–96; MR **15**, 578; **18**, 837], the author derived from the equations of equilibrium the Beltrami-Mitchell compatibility equations in tensor notation. Here he derives them again by starting from the compatibility equations in terms of strains. His result differs from that of M. Brdička [Czechoslovak J. Phys. **3** (1953), 36–52; MR **15**, 578] in taking into account body forces which Brdička neglected.

T. Leser (Aberdeen, Md.)

1462:

Ferrarese, Giorgio. Sugli spostamenti regolari dei solidi tubolari. Rend. Mat. e Appl. (5) **16** (1957), 352–362.

This paper contains an attempt to derive from the three-dimensional theory of finite elastic strain the classical formulae for the bending and twisting of thin rods. [Cf. the treatment by Hay, Trans. Amer. Math. Soc. **51** (1942), 65–102; MR **3**, 287]. Under very restrictive assumptions regarding the three-dimensional deformation, the author shows that the Kirchhoff formulae are exact to within a second order neighborhood of a point on the directrix.

C. Truesdell (Bloomington, Ind.)

1463:

Kane, T. R. Reflection of dilatational waves at the edge of a plate. J. Appl. Mech. **24** (1957), 219–227.

A theory which approximates the motion of a plane elastic slab or plate with middle surface moving parallel to itself is applied to wave propagation. Slab surfaces are $z=\pm h$; displacements in the plane are assumed independent of z, and proportional to z in the z direction. Phase velocities of the resulting three modes are compared with the exact three-dimensional theory. Reflection of plane waves at a plane free surface is investigated and various reflection phenomena are detailed, including edge vibration decreasing exponentially away from the edge and non-decreasing vibration, in addition to reflected waves. The results are compared with a more approximate analysis based on generalized plane stress.

E. H. Lee (Providence, R.I.)

1464:

***Flügge, Wilhelm. Statik und Dynamik der Schalen. 2te, neubearbeitete Aufl.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957. viii+286 pp. DM 28.50.

The first edition of this well-known work appeared in 1934 [Springer, Berlin]. The major portions of the first edition have been taken over into the second edition. This means that, as before, the book deals to a considerable extent with linear membrane theory of shells of revolution, with the linear theory of circular cylindrical shells and with linear bending theory for symmetrical deformations of shells of revolution. Some consideration is given to classical elastic stability theory of circular cylindrical shells and spherical shells and to vibrations of shells. New material includes consideration of the physical effect of negative total curvature of shell surfaces and membrane analysis of hyperbolic paraboloidal and of conoidal shells. A bibliography at the end of the book refers to some of the literature which has come into being since the appearance of the first edition. A distinct virtue of the book consists in the frequent physical interpretations of analytical results.

E. Reissner (Cambridge, Mass.)

1465:

***Kollbrunner, Curt F.; und Meister, Martin. Ausbeulen: Theorie und Berechnung von Blechen.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. xi+344 pp. DM 42.00.

This is a quite comprehensive treatment of the buckling of rectangular plates, principally devoted to the classical problem but including also considerable material on the effect of plasticity and stiffeners. The authors state that they are more concerned with practical applications than with theory, but they devote the first eighty pages to an excellent presentation of classical plate bending theory, membrane theory (two dimensional elasticity), and three methods (direct integration, and energy and numerical methods) for applying these theories to buckling, which they treat as a mixed bending and membrane problem. The next hundred and ten pages are devoted to classical solutions for a great variety of loading cases. Most of the remainder of the book is divided between theoretical discussions of plastic buckling of plates and the effect of stiffeners, and some test results. In a few pages at the end the authors consider the pseudo-stability case where lateral disturbances or defects are present, and the post-buckling case. The book is very well illustrated.

Most of the book describes the work of the authors and of other Swiss and German investigators, with comparatively little reference to American and English work and none to modern Russian sources.

L. H. Donnell (Chicago, Ill.)

1466:

Csonka, P. Elasticity theory of plane plates of uniform thickness. Period. Polytech. Engrg. **1** (1957), 103–120.

The author outlines a solution, satisfying elasticity theory, for small displacements of thick plates under any surface loading. The loads are separated into symmetric and antisymmetric, normal and tangential components, and each of these is expanded in infinite series of functions which might be described as generalized harmonic functions. A particular solution of elasticity conditions is developed for the loading represented by each of these

functions. Displacements and stresses corresponding to each particular solution are first expressed as infinite series with terms involving higher and higher derivatives of a function of the two mid-surface coordinates, multiplied by power functions of the thickness coordinate; it is shown, however, that for the class of functions in question these latter series can be reduced to closed form. Some solutions for zero loading on the lateral surfaces are developed in similar fashion, which are to be combined with the above solutions to satisfy edge boundary conditions; it is not at all clear, however, that these are sufficiently general to accomplish this purpose.

Application to a specific problem is promised in a later paper. Evaluation of the promise of this, as well as of analogous solutions such as Green's general solution for edge conditions or Donnell and Lee's general solution for surface loading, will have to await such demonstrations of application. *L. H. Donnell* (Chicago, Ill.)

1467:

Nariboli, G. A. Mixed boundary value problems for rectilinear plates. II. Arch. Mech. Stos. **10** (1958), 129–142. (Polish and Russian summaries)

The author here extends the results of his earlier paper [Arch. Mech. Stos. **9** (1957), 507–524; MR **19**, 1107] to the problems of buckling and free vibrations of some rectilinear plates.

By use of the analogy that the Laplacian of deflection of a supported rectilinear plate corresponds to the stream function due to a vortex in a cylinder of the same cross-section, the plate problem is reduced to a two step harmonic problem. Here this solution is used to obtain the deflection due to unit moment applied normal to the edge. The solution of total deflection due to distributed edge moments along the boundary of a supported plate is then represented in an integral form. The condition that the slope must vanish on the clamped part leads to a homogeneous Fredholm integral equation for moment distribution. This is reduced to the solution of an infinite set of algebraic equations whose solubility condition leads to the characteristic equation as the vanishing of an infinite determinant.

The cases of right-angled isosceles triangle, right-angled triangle with 30° angle and equilateral triangle are treated in detail. Numerical values of the first eigen-value are obtained in each case, and compared with the values under supported boundary conditions. The results show that for the same boundary, the rise in the parameter giving the least buckling load is higher than that giving fundamental frequency and the rise increases with the decrease in angle. *B. R. Seth* (Kharagpur)

1468:

Bychawski, Z.; and Piszczek, K. Pseudo-plane state of shrinkage distortion in a non-homogeneous circular cylinder. Arch. Mech. Stos. **10** (1958), 211–231. (Polish and Russian summaries)

Attempt is made to establish equations in exponential form describing the process of shrinkage as function of time, taking physical parameters into consideration. Assuming the axial strain to be independent of spatial coordinates, the equilibrium equation for a long non-homogeneous cylinder is solved by a perturbation method, confined to the first approximation.

The method is also employed in the case of a heterogeneous cylinder characterized not only by different properties in different regions but also by physical surfaces separating these regions. The inner cylinder (steel) is subjected to elastic distortion, i.e., is in a state of initial strain due to prestressing, while the outer one (concrete) is in a state of shrinkage distortion.

Illustrations by numerical examples are given in both cases. *S. C. Das* (Chandernagore)

1469:

Stroh, A. N. Dislocations and cracks in anisotropic elasticity. Phil. Mag. (8) **3** (1958), 625–646.

Following the treatment of Eshelby et al. [Acta Met. **1** (1953), 251–259] for the theory of anisotropic elasticity for a three dimensional state of stress in which the stress is independent of one of the Cartesian coordinates, fuller discussions on stresses due to dislocations are given. The energy of interaction of two dislocations is found by Cottrell's method [Progress in metal physics I, Interscience, New York, 1949, 77–126] of forming one dislocation in the presence of the other, and the stresses due to a wall of parallel dislocations are obtained. Stresses are also considered round a crack under arbitrary non-uniform applied stress. The results are presented in analytically simple forms. *S. C. Das* (Chandernagore)

1470:

Ambarcumyan, S. A. On the theory of bending of anisotropic plates. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1958**, no. 5, 69–77. (Russian)

Bending of anisotropic plates with one plane of elastic symmetry coinciding with the middle plane xy of the plate is considered, neglecting σ_z and assuming $\varepsilon_z=0$ and an arbitrary law $f(z)$ of the variability of τ_{zx} and τ_{zy} with z. For an orthotropic plate with principal elastic directions parallel to the coordinate axes, a differential equation of the sixth order for the strain function is obtained. Solution of a problem concerning a square, simply supported, transversely isotropic plate shows that the magnitude of the displacement and stress slightly depends on the form of $f(z)$. Thus, for $f(z)$ constant, quadratic or of the fourth degree, the deflections differ less than 10%. The classical solution, however, differs up to 200%. Again, for a thick square isotropic plate the difference between the classical and rigorous solutions amounts to 35%, while the difference between the latter and that of the author is less than 6%. *J. Nowinski* (Madison, Wis.)

1471:

Satô, Yasuo; and Yamaguchi, Rinzo. Vibration of a building upon the elastic foundation. Bull. Earthquake Res. Inst. Tokyo **35** (1957), 545–565. (Japanese summary)

This discussion includes the effect of reactions from the building in modifying the ground motion, rather than treating the ground motion as inexorable, as had been usual. Three modes of motion are discussed: vertical, horizontal, and rotatory about a horizontal axis. For each of these, resonances are obtained, even though the building models used in discussing the first and third of these modes are rigid. *A. Blake* (Ann Arbor, Mich.)

1472:

Volkov, A. N. Die Schwingung einer Zylinder-Schale in der Strömung einer idealen Flüssigkeit. Apl. Mat. **3** (1958), 161–169. (Czech. Russian and German summaries)

The effect of the mass of a surrounding acoustic medium on the frequency of the free vibrations of a sub-

merged thin cylindrical shell is investigated under various simplifying assumptions. The author is apparently unaware of the numerous papers dealing with various aspects of free and forced vibrations in submerged shells, published in this country since 1952, with results that go far beyond those obtained by the author.

A. M. Freudenthal (New York, N.Y.)

1473:

Sundström, Erik. Creep buckling of cylindrical shells. Kungl. Tekn. Högsk. Handl. Stockholm no. **115** (1957), 34 pp.

This paper deals with the problem of creep buckling of long cylindrical sandwich (and solid) shells under the simultaneous action of an external pressure and axial forces. For the creep law the Norton formula, modified by Odqvist, is used which relates the strain rates to the stress rates and the stresses by equations containing powers of the second scalar invariant of the stress deviation tensor. Because of the non-linearity of this phenomenological relation a rigorous solution could not be obtained; but by developing the deflection of the shell into harmonics and taking into account only the first term, called fundamental mode, an approximation has been found. The neglecting of the higher modes is justified to some extent by the creep law itself. *W. Schumann* (Zürich)

1474:

Capriz, Gianfranco. Alcune osservazioni su problemi di instabilità delle travi elastiche. Rend. Mat. e Appl. (5) **16** (1957), 23–42.

By a method analogous to Signorini's expansion for the general non-linear theory of elasticity, the author studies the instability of a bar in compression according to the non-linear Bernoulli-Euler theory. He perceives that the undetermined amplitude given by the classical first approximation can be rendered definite by use of conditions of compatibility for the higher approximations. {The series he begins to calculate in this way was obtained in full by Euler; what is new here is the connection with Signorini's conditions of compatibility.} The author then sets up the corresponding problem for a built-in beam subject to a point load and a torsional couple, but the result is negative. If there is a stable solution, this method does not yield it. *C. Truesdell* (Bloomington, Ind.)

1475:

***Ewing, W. Maurice; Jardetzky, Wenceslas S.; and Press, Frank. Elastic waves in layered media.** Lamont Geological Observatory Contribution No. 189. McGraw-Hill Series in the Geological Sciences. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1957. xi+380 pp. $10.00.

Chapter 1 develops the equations of motion for small disturbances in a perfectly isotropic medium, derives the P and S wave velocity formulae, and discusses solutions of the one-, two- and three-dimensional wave equation.

Chapter 2 considers wave problems in a half-space of perfectly elastic isotropic homogeneous material. The problems include reflection and refraction of P and S waves, partitioning of energy at the boundary, reflection at critical angles, and Rayleigh waves, and there is a detailed account of Lamb's classic paper on disturbances due to impulsive forces applied at line or point sources on the boundary. Recent work of authors such as Nakano, Sobolev, Lapwood and Newlands is also discussed. The theoretical results are illustrated in seismological contexts, and are compared with the results of seismic model experiments.

Chapter 3 considers seismic wave propagation in two semi-infinite media in contact at a plane boundary. The theory is applied to near-earthquake problems and seismic prospecting. Work of Sommerfeld on electromagnetic waves is adapted to the seismic case. There is an account of generalized Rayleigh waves, and discussions are given of work of authors such as Stoneley, Sezawa, Cagniard and Scholte.

Chapter 4, on a layered half-space, considers Love waves and a variety of cases of more complicated layering. The necessary theory of dispersion and group velocity is given using Fourier transform methods. Waves in fluid and solid media are considered, and dispersion curves drawn for a large number of cases. The important contributions of Jeffreys and Stoneley are given, and there is a section on air-coupled Rayleigh waves. The theory is applied to the problem of the surface structure of the Earth.

In Chapter 5, the theory is extended to include such effects as disturbances in gravity, curvature of boundaries, and viscosity, to the extent that these are significant in seismology.

Chapter 6 on plates and cylinders is of less immediate relevance to seismology, and contains material of interest to the general worker on elasticity vibration problems, and on the theory of sound.

Chapter 7 treats media in which the wave velocities are continuously variable.

There is an appendix on the method of steepest descents and on Rayleigh's principle. References are good and up-to-date.

The book is specially valuable because of the summary it contains of the considerable and important post-war contributions of the Lamont Geological Observatory to surface wave problems in seismology, and related matters.

K. E. Bullen (Sydney)

1476:

***Бреховских, Л. М. [Brehovskih, L. M.] Волны в слоистых средах. [Waves in layered media.]** Izdat. Akad. Nauk SSSR, Moscow, 1957. 502 pp. 26.10 rubles.

In the same year 1957 when Brekhovskikh's book was published in the USSR, an excellent book by Ewing, Jardetzky, and Press, "Elastic waves in layered media" [reviewed above], was published in the USA. Readers interested in the subject must be familiar with the American book; hence a comparison of the two books may serve the purpose best.

In the American book the authors assembled material not only on the transmission of elastic waves in layered media but also basic information on transmission of elastic waves, plane and spherical, in general. The book gives solutions of wave equations for infinite, semi-infinite and layered media, waves being propagated from plane, line and point sources. It also contains considerable material on plates and cylinders. Although the mathematical treatment of electromagnetic waves, water waves, and shock waves is similar to the methods used for elastic waves, these other waves are mentioned only in a brief reference. The theoretical treatment and the experimental methods used for elastic waves were up to now kept apart. In this book for the first time theoretical and experimental methods interplay whenever it is possible.

Brekhvoskikh's book contains practically the same material as the American counterpart plus the treatment of electromagnetic and water waves. The author omitted

entirely an introductory chapter with an outline of the theory of motion in elastic media and derivation of the equations of motion; the book starts directly from the reflection and refraction of waves. The author considers also both theory and experiment. Brekhovskikh's book is not nearly so well organized as the American one and, as many other Soviet books, contains no index. The book has the following chapters: I. Plane waves in layers; II. Certain applications of the theory of propagation of plane waves in layered media; III. Plane waves in layered non-homogeneous media; IV. Reflection and refraction of spherical waves; V. Propagation of waves in layers; VI. Field from point source in a layered non-homogeneous medium. *T. Leser* (Aberdeen, Md.)

1477:

Kästner, Siegfried. Zur Theorie der Relaxation. I. Die thermodynamischen Grundlagen und ihre mathematische Lösung. Ann. Physik (7) **1** (1958), 377–399.

The paper refers to relaxation behavior of homogeneous systems. Deviations from thermodynamical equilibrium are supposed to be small; this is equivalent to the assumption of linearity and guarantees validity of principle of superposition and theory of irreversible processes. On this basis the thermodynamical equations are established, in tensorial form, first for "pure" relaxation, then for the case of internal friction, and finally for the general case including both. Solutions are discussed for steady state and for transient behavior in terms of spectral tensors which are generalizations of the well-known relaxation and retardation spectra. Applications to mechanical and dielectric relaxation are discussed.

B. Gross (Rio de Janeiro)

1478:

Olszak, Wacław. The inversion mapping as applied in the theory of plasticity. Arch. Mech. Stos. **10** (1958), 417–440. (Polish and Russian summaries)

The usefulness of conformal mapping in solving elastic problems is well known. As first pointed out by Galin [Prikl. Mat. Meh. **10** (1946), 367–386; MR **8**, 241] (rather surprisingly, the author does not give this reference) conformal mapping is also useful in elastic-plastic problems where the resulting plastic stress function happens to be biharmonic.

Here the author considers the particular conformal mapping of "inversion" which consists of a reflection in the unit circle. With suitable choice of pole and units, such a mapping transforms an arbitrary eccentric annular cylinder into a concentric one.

In order to avoid direct consideration of a multiply connected region, the cylinder is cut along the axis of symmetry on the side towards the pole. The eccentric cylinder is loaded with uniform normal pressures p and q along the internal and external boundaries and a resultant normal force P along the cut. Under these conditions, the elastic problem for the mapped region is rotationally symmetric and easily solved. The mapping then provides the elastic solution for the original eccentric cylinder.

The fully plastic and elastic-plastic problems can be solved by the same technique provided that the yield strength of the material is inversely proportional to the distance from the pole. The resulting mapped problem will still be rotationally symmetric and the stress function is still biharmonic.

For the fully elastic problem. displacements can be computed and P determined for a given p and q so that the displacements are single valued, thus providing the solution for the whole cylinder. However, for the elastic-plastic cylinder, the force P is determined by the condition that hydrostatic stress remains finite in the elastic region. Therefore, the solution obtained is valid only for restricted loadings of slit cylinders and does not, in general, apply to the whole cylinder. The author points out this fact and suggests that the whole cylinder solution may be obtained by considering further terms in the stress function. *P. G. Hodge, Jr.* (Chicago, Ill.)

1479:

Olszak, Wacław; and Mróz, Zenon. Note on the completeness of the elastic-plastic solution to certain boundary value problem for the eccentric ring. Arch. Mech. Stos. **10** (1958), 441–444. (Polish and Russian summaries)

The authors show that the form of the plastic stress function assumed in the paper reviewed above is the most general rotationally symmetric one possible. This would seem to imply that the solution of the whole cylinder problem is not rotationally symmetric in the mapped plane. *P. G. Hodge, Jr.* (Chicago, Ill.)

1480:

Kadaševič, Yu. I.; and Novožilov, V. V. The theory of plasticity with the Bauschinger effect taken into account. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 586–588. (Russian)

General, quasi-linear stress-strain relations of an isotropic plastic medium are proposed that represent the Bauschinger effect on unloading by making the plasticity condition a function not only of a stress-invariant alone, but also on the permanent strain. The approach is similar to that considered, in more detail, by Prager in several papers on the rigid-workhardening materials.

A. M. Freudenthal (New York, N.Y.)

1481:

Košelev, A. I. On some problems of the theory of plasticity. Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. **12** (1957), no. 19, 20–29. (Russian. English summary)

"This article is devoted to the problem of elastico-plastic torsion of bars. The existence of weak solution is proved under certain assumptions. It is shown that the method of elastico-plastic solutions can diverge, in spite of existence of solution. The first boundary problem for elastico-plastic bodies is also considered. In this case the existence of weak solution is proved." (Author's summary.) *H. G. Hopkins* (Sevenoaks)

1482:

***Талыпов, Г. Б. Приближенная теория сварочных деформаций и напряжений. [Talypov, G. B. Approximative theory of welding deformations and strains.]** Izdat. Leningradsk. Univ., 1957. 206 pp. (1 insert) 9.00 rubles.

1483:

***Ржаницын, А. Р. Некоторые вопросы механики систем, деформирующихся во времени. [Ržanicyn, A. R. Some questions in the mechanics of systems undergoing deformation.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1949. 252 pp. (1 insert). 10.15 rubles.

The author develops methods for calculation of non-elastic structures, applying the methods to practically important cases of variable stress-states.

In Ch. I it is demonstrated how the time-variable loads can be taken into account in the theory of elasto-plastic systems. In Chs. II–VI the linear theory of deformation of time-dependent materials based on the simplest known differential stress-strain relations is discussed. In Ch. VII the vibrations of a system subjected to time dependent deformation, according to the above mentioned simple laws, are investigated. Chs. VIII–X are devoted to improving the linear stress-strain relations, with reference to the general law of deformation given in integral form, as well as to investigations of problems of stability and vibrations under this general law. Finally, Ch. XI treats non-linear deformations in the case of large stresses and strains. *T. P. Anđelić* (Belgrade)

1484:

Aksel'rad, È. L. On thermal deformations of inhomogeneous shells. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1958**, no. 8, 48–52. (Russian)

Fictitious body force and surface tractions corresponding to a change of temperature field in an isotropic heterogeneous shell with arbitrarily variable coefficient of thermal expansion and elastic coefficients are determined. Now, according to the author's previous paper the problem of a transversely (i.e. along the thickness) nonhomogeneous shell acted upon by external load is equivalent to that for a homogeneous shell. It results, that the thermo-elastic problem for a transversely nonhomogeneous shell is equivalent to a non-thermal problem for a homogeneous shell.

J. Nowinski (Madison, Wis.)

STRUCTURE OF MATTER

See 1526, 1559.

FLUID MECHANICS, ACOUSTICS

See also 1056, 1422, 1451, 1472.

1485:

***Emmons, Howard W. Theoretical aerothermodynamics.** 1958 Heat transfer and fluid mechanics institute, held at University of California, Berkeley, Calif., June, 1958: preprints of papers, pp. 1–14. Stanford University Press, Stanford, Calif., 1958. viii+264 pp. $7.50.

This is a very brief review of the subject of fluid mechanics with chemical reactions. After writing down the equations of motion (with a few remarks concerning the effects neglected) the author discusses two simple examples: (1) nozzle flow without friction or heat transfer but with a simple reaction involving only two components, and (2) combustion in the boundary layer of a flat plate with one-step reaction. It is concluded that "the analytical solution of combustion problems using simplified chemistry is ready for extensive development."

W. R. Sears (Ithaca, N.Y.)

1486:

Broadwell, James E. A simple model of the non-equilibrium dissociation of a gas in Couette and boundary-layer flows. J. Fluid Mech. **4** (1958), 113–139.

This paper calculates the effects of dissociation of a diatomic gas on heat transfer in three cases: (1) in a layer of gas at rest between two walls at different temperatures, (2) in Couette flow, and (3) in a laminar boundary layer on a flat plate. By assuming that the reaction rate is a linear function of temperature and weight fraction of atoms, the problem in each case is reduced to a differential system for these two variables for both catalytic and non-catalytic wall conditions. *Y. H. Kuo* (Peking)

1487:

Freeman, N. C. Non-equilibrium flow of an ideal dissociating gas. J. Fluid Mech. **4** (1958), 407–425.

This paper investigates the behavior of a dissociating diatomic gas in a flow through a strong normal shock and past a bluff body. The calculation is carried out for a special form of reaction rate. The results indicate that for both cases changes from the undissociated values of all physical quantities are considerable.

Y. H. Kuo (Peking)

1488:

Gheorghiev, Gh. Quelques aspects géométriques du mouvement permanent d'un fluide idéal. II. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **2** (1956), 69–84. (Romanian. Russian and French summaries)

Continuing the author's study in Rev. Univ. "Al. I. Cuza" Inst Politehn. Iaşi. **2** (1955), 43–64 [MR **18**, 616], on barotropic fluids, certain compatibility conditions are established and theorems derived for the case that the motion of the fluid is uniform on every streamline. The method is also useful for the general study of the geometry of the streamlines of these "motions B". The second part of the paper establishes the necessary and sufficient condition that a family of surfaces and a congruence of curves on these surfaces are, respectively, surfaces S_B and the streamlines of the motions B.

D. J. Struik (Cambridge, Mass.)

1489:

Popescu, I. L. Sur le mouvement non-permanent d'un fluide à travers une grille de profils. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **9** (1957), 121–137. (Romanian. Russian and French summaries)

According to the summary, the author has studied the extension of the work of Sedov on thin wings in unsteady motion [Ploskie zadači gidrodinamiki i aèrodinamiki, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950; MR **19**, 346] to the case of a cascade. Perfect incompressible flow is assumed, and the problem is reduced to solution of an integral equation which is solved by operational methods. The particular case of harmonic oscillations is treated. The calculation of forces will be given in a second paper. *W. R. Sears* (Ithaca, N.Y.)

1490:

Murray, James D. Non-uniform shear flow past cylinders. Quart. J. Mech. Appl. Math. **10** (1957), 406–424.

In this paper, a general method is described whereby a unique solution can be obtained for two types of shear flow past a general cylinder. The two shear distributions in the undisturbed stream considered are one that is a linear combination of hyperbolic cosine and sine and the other, a linear combination of cosine and cosine. Two special examples are cited. *Y. H. Kuo* (Peking)

1491:

Cox, A. D.; and Clayden, W. A. Cavitating flow about a wedge at incidence. J. Fluid Mech. **3** (1958), 615–637. (1 plate)

A major difficulty in the theory of steady cavitating

flow past a wedge at incidence is to set up a realistic model of the flow. If the stagnation point is assumed to occur at the vertex of the wedge the lengths of the sides of the wedge cannot be prescribed arbitrarily. On the other hand if the stagnation point occurs on one side then the realistic subsidiary cavity at the vertex can only extend to infinity. If the dimensions and orientation of the wedge preclude this possibility the only alternative model is an unrealistic one with a re-entrant jet. Such a model implies that some of the fluid will be moving in the wrong direction at infinity upstream of the wedge.

The authors calculate the flow past a wedge at zero cavitation number using the model and show that the rate of loss of momentum in the entrant jet is of second order when the angle of incidence is small, so that in spite of the apparently unrealistic nature of the model it should give accurate determination of the lift drag and moment slopes at zero incidence. The authors also carried out some experiments on cavitating flow past wedges at incidence which confirmed the conclusions of the theory.

K. Stewartson (Durham)

1492:

Gupta, Subhas Chandra. Slow broad side motion of a flat plate in a viscous liquid. Z. Angew. Math. Phys. **8** (1957), 257–261.

The slow motion of incompressible viscous liquid caused by the steady broadside motion of a finite flat plate is expressed in terms of a single harmonic function. The known motion caused by a circular plate is derived as an example.

W. R. Dean (London)

1493:

***Bhatnagar, P. L.; and Verma, P. D. On superposable flows.** Proceedings of the Second Congress on Theoretical and Applied Mechanics, New Delhi, October, 1956, pp. 141–152. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur.

It is shown that the necessary and sufficient condition that two viscous, incompressible flows be superposable is

$$\nabla\times(\omega_2\times q_1)+\nabla\times(\omega_1\times q_2)=0,$$

where q and ω are velocity and vorticity vectors respectively and suffixes 1 and 2 denote the flows whose superposability is being investigated.

This result is applied to seek axi-symmetric flows which are superposable on certain simple rotational flow fields. For the simple flows investigated, only one, that of velocity field due to a single vortex, had a flow field which was superposable on it.

J. J. Mahony (Sydney)

1494:

Kotlyar, Ya. M. Flow of a viscous gas in a narrow space between two coaxial cylinders. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 10, 12–18. (Russian)

Gas is injected through equally-spaced orifices in the outer cylinder. Under assumptions appropriate to slow flow in a thin layer, the problem is reduced to one of plane potential flow, and solved using hydrodynamic sources and the method of images.

M. D. Van Dyke (Paris)

1495:

Săvulescu, St. Application de la méthode des profils type de vitesses et de températures au calcul de la couche limite sur un corps de révolution. Acad. R. P. Romîne. Stud. Cerc. Mec. Apl. **8** (1957), 975–982. (Romanian. Russian and French summaries)

Dans cet article, l'auteur étend sa méthode de calcul des caractéristiques de la couche limite bidimensionnelle au cas tridimensionnel des corps de révolution. On montre que, en utilisant la variable généralisée

$$\eta_{(1)}=\frac{\Psi_r}{\Psi_{r\delta\Delta}}=\frac{\int_0^{\delta}\rho u(R\pm\cos\alpha)dr}{\int_0^{\delta(\Delta)}\rho u(R\pm r\cos\alpha)dr}$$

les vitesses et les énergies adimensionnelles peuvent se mettre sous la forme type

$$u_a=\eta^n,\quad E_0\cong\eta_1^{n_1}$$

où n, n_1 diminuent légèrement en égard du cas bidimensionnel.

Résumé de l'auteur

1496:

Lighthill, M. J. On displacement thickness. J. Fluid Mech. **4** (1958), 383–392.

The author gives a new account of displacement-thickness in the boundary-layer theory. The principal expression for displacement-thickness δ_1 in a three-dimensional boundary-layer is shown to be

$$\delta_1=\delta_x-\frac{1}{Uh_y}\frac{\partial}{\partial y}\int_0^x\delta_y dx,$$

where x and y are "external flow" coordinates on a surface, namely, x=const and y=const, give respectively equipotential and streamlines of the external flow; $h_y dy$, the distance between (x, y) and $(x, y+dy)$; δ_x and δ_y, the streamwise and transverse "volume-flow thicknesses" and U, the velocity just outside the boundary layer.

Y. H. Kuo (Peking)

1497:

Walz, Alfred. Näherungstheorien für die Berechnung von Strömungsgrenzschichten. Z. Angew. Math. Phys. **9b** (1958), 695–709.

An approximate calculation procedure based on integral methods is developed for compressible turbulent boundary layers with heat transfer at the wall. Generalizations of the well-known empirical laws for skin friction and dissipation in the incompressible case are used, and it is suggested that these are valid up to about Mach number 10 under arbitrary heat transfer conditions. Universal functions have been evaluated so that practical calculations may be performed rapidly. Comparisons of theoretical results with experiments show good agreement. A brief examination is also made of the application of integral methods to the complete Navier Stokes equations in cases when the boundary-layer equations are not adequate.

D. W. Dunn (Ottawa, Ont.)

1498:

Görtler, Henry; und Witting, Hermann. Einige laminare Grenzschichtströmungen, berechnet mittels einer neuen Reihenmethode. Z. Angew. Math. Phys. **9b** (1958), 293–306.

The formally exact series solution for a boundary layer with arbitrary free-stream velocity $U(x)$ developed previously by Görtler [J. Math. Mech. **6** (1957), 1–66; MR **18**, 843] is applied to the numerical evaluation of several special cases: (i) $U(x)=U_0(1+x/L)^{-n}$ for several values of $n\geqq 0$; (ii) $U(x)=U_0(1-x/L)^n$ for several values of $n\geqq -1$; (iii) $U(x)=U_0\cos(x/L)$; (iv) $U(x)=u_1R\sin(x/R)$ (the potential-flow velocity distribution around a circular cylinder). The new method requires only a short and simple numerical calculation, involving the application of tables of certain universal functions.

D. W. Dunn (Ottawa, Ont.)

1499:
Lun'kin, Yu. P. Boundary layer equations and their boundary conditions in the case of motion of a weakly rarefied gas with supersonic speeds. Prikl. Mat. Meh. 21 (1957), 597–605. (Russian)

Starting from the equations proposed by Chapman for flow of a slightly rarefied gas (and correcting an error), the author derives the counterpart of Prandtl's boundary layer equations. The pressure is found to be no longer constant across the boundary layer. Corresponding boundary conditions at the wall are found in terms of an accomodation coefficient by assuming that the third approximation to the distribution function is valid there. The physical range of applicability is discussed, and a laborious iterative scheme proposed for solving the equations. *M. D. Van Dyke* (Paris)

1500:
Bass, J. On the mathematical structure of turbulence. Rev. Mod. Phys. 30 (1958), 1084–1085.

The author considers the possibility of representing turbulence by explicit functions that exhibit the required properties of irregularity. Taking as a simple example the equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\frac{1}{2}\frac{\partial^2 u}{\partial x^2} \tag{1}$$

used by Burgers as a model for turbulence, he shows that the almost periodic functions that satisfy the equation do not have these properties, in particular that of zero autocorrelation for large time intervals. The desired irregularity may be introduced by multiplying an almost periodic function by a function $Y(t)$ which alternates between the values, $+1$ and -1, in a suitably irregular way. In particular the partial differential equation (1) admits the solution

$$u(x, t)=\int_0^\infty Y(t+\omega)\phi(\omega, x)d\omega,$$

where $\phi(\omega, x)$ is defined for $\omega\geqq 0$ and has a finite integral. This type of function is capable of representing turbulence. *A. A. Townsend* (Cambridge, England)

1501:
Mager, Artur. Transformation of the compressible turbulent boundary layer. J. Aero. Sci. 25 (1958), 305–311.

This is identical with the paper in the Proceedings of Heat Transfer and Fluid Mech. Inst., Cal. Tech., Pasadena, 1957, pp. 85–98, Stanford Univ. Press [MR 19, 604.]
D. A. Spence (Farnborough)

1502:
Culick, Fred E. C.; and Hill, Jacques A. F. A turbulent analog of the Stewartson-Illingworth transformation. J. Aero. Sci. 25 (1958), 259–262.

A transformation similar to those of Stewartson and Illingworth is applied to the momentum thickness and external velocity distribution of the turbulent boundary layer. This reduces the integral momentum equation to the same form as in incompressible flow, provided (i) the form parameter H is taken as $(H_i+1)(T_0/T_e)-1$, where H_i is the corresponding incompressible value (T_0 and T_e are the total and ambient temperatures in the external stream); (ii) the x coordinate is transformed in accordance with the accepted intermediate-temperature relation between compressible and incompressible skin friction coefficients. Experimental evidence from accelerated flows is shown to justify (i), for Mach numbers up to 5 at low heat transfer rates. An expression is obtained which yields the momentum thickness in terms of simple quadratures. A similar result has also been given by Mager [see review above].
D. A. Spence (Farnborough)

1503:
Sauer, R. Einführung in den Kalkül der Distributionstheorie mit Anwendungen auf Anfangswertprobleme in der Gasdynamik. Abh. Math. Sem. Univ. Hamburg 22 (1958), 50–70.

L'auteur rappelle d'abord quelques formules de la théorie des distributions. Il les applique à des problèmes aux limites (problème d'une aile portante mince presque plane dans un courant supersonique), en employant la méthode habituelle: si on considère la solution cherchée, et qu'on la remplace par 0 dans certaines régions de l'espace-temps, on obtient une nouvelle fonction, satisfaisant à des équations modifiées, où les anciennes conditions aux limites apparaissent maintenant dans le second membre.
L. Schwartz (Paris)

1504:
***von Mises, Richard. Mathematical theory of compressible fluid flow. Completed by H. Geiringer and G. S. S. Ludford.** Applied mathematics and mechanics. Vol. 3. Academic Press, Inc., New York, N.Y., 1958. xiii+514 pp. $15.00.

The first three chapters were written by Professor von Mises; the last two were added after his death by H. Geiringer and G. S. S. Ludford. Chapters I (Introduction) and II (General theorems) formulate the partial differential equations of compressible flow with and without consideration of viscosity and heat conduction; develop the acoustic approximation to almost uniform flow; derive Kelvin's theorem and Helmholtz's vortex theorems; give a heuristic discussion of the theory of characteristics for hyperbolic systems, including a method for numerical integration; and develop Riemann's method for linear hyperbolic partial differential equations. Chapter III (One-dimensional flow) develops various exact linearized forms for the equations of isentropic flow; considers simple waves; derives the general form of the solutions of the simplest cases of the corresponding Euler-Poisson-Darboux equation; develops shock wave theory from one-dimensional shock layer in a viscous heat-conducting gas; and considers various shock wave reflection and interaction problems. Chapter IV (Plane steady potential flow) discusses the usual exact linearizations by transformation to the hodograph plane; considers simple waves, limit lines, and branch lines; discusses the Karman-Tsien approximation; and explains Chaplygin's method for subsonic jets. Chapter V (Integration theory and shocks) describes Lighthill's and Bergman's extensions of Chaplygin's method, discusses steady shock theory and problems for plane flow; and concludes with a discussion of transonic flow based on one of the senior author's papers.

Fullest discussion has been given to those mathematical techniques that lead most directly and explicitly to some exact solutions of the partial differential equations, such as the methods for modifying incompressible hodograph solutions into compressible hodograph solutions; solution of special cases of the Euler-Poisson-Darboux equation; or examination of special examples. Material of a more purely abstract existence-theoretic nature, such as Schiffman's or Bers' proofs of the existence of purely subsonic potential flow about prescribed closed bodies, is presented by summarizing results without discussing methods. It should also be mentioned that the

earlier chapters seem to give more of an intuitive feeling for the subject matter than the later ones. For all chapters there are excellent elaborate sets of notes which extend the discussion of the text and frequently give good short summaries of the numerous papers referred to. For the beginning student of compressible flow theory the book provides a valuable guide to a substantial part of the technical literature. For the more experienced reader it also provides stimulating reading and directs attention to some generalizations, extensions, or finer points of the theory. *J. H. Giese* (Aberdeen, Md.)

1505:

Patraulea, N. N. Profils aérodynamiques à volet fluide. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **9** (1957), 451–455. (Romanian. Russian and French summaries)

Patraulea derives an analytical method for calculating the aerodynamic profiles with the jet flap. The conditions at infinity are based on the physical aspects of the problem and are approximated so as to make the problem linear. The small perturbation process is applied and the problem is reduced to an integral linear equation. No numerical example is calculated.

M. Z. v. Krzywoblocki (Urbana, Ill.)

1506:

Meyer, Richard E. On the structure of supersonic flow. Z. Angew. Math. Phys. **9b** (1958), 454–461.

A survey of some general properties of steady plane shock-free supersonic flow that are invariant and so valid in the large. With the aid of characteristic variables, consideration is given to the focussing of Mach waves, the relation between wave fronts and the analytic continuation of a finite region of flow, and the properties of branch lines and limit lines. *M. D. Van Dyke* (Paris)

1507:

Keller, Joseph B. Acoustic torques and forces on disks. J. Acoust. Soc. Amer. **29** (1957), 1085–1090.

By expressing the pressure in an acoustic field to terms of second order in the harmonic velocity potential the Levine-Schwinger expressions for time average force and torque on an immersed immobile "disk" are derived. Plotted curves of torque vs. angle of incidence for a long strip (width $2a$) and a circular disk (radius a) at $ka=5$ oscillate between positive (unstable) and negative (stable) values.

W. W. Soroka (Berkeley, Calif.)

1508:

Naugol'nyh, K. A. On the absorption of sound waves of finite amplitude. Akust. Ž. **4** (1958), 115–124. (Russian)

Review article.

1509:

***Ree, Taikyue; and Eyring, Henry. The relaxation theory of transport phenomena.** Rheology: theory and applications. Edited by F. R. Eirich. Vol. 2, pp. 83–144. Academic Press, Inc., New York, 1958. $18.00.

The article represents an attempt to discuss phenomena of flow in fluids, in rheological media and in solids in terms of a generalized rate theory based on the concept of relative motion (jumps) of molecular flow units with respect to each other on both sides of a shear plane, in the course of which the acting shear stress is released. The underlying flow equation in all media is therefore the hyperbolic sine relation between the flow rate $\dot{s}$ and the shear stress f proposed by Prandtl (1928) and redefined by Eyring (1936) in terms of a rate process of formation of "holes" $\dot{s}=\text{const}\sinh Cf$, where the constant parameters are functions of the size and the distances of flow units, the absolute temperature and the free activation energy of the considered flow process. This relation is generalized by the superposition of a series of such processes with different parameters, as well as by the consideration of "entanglement" between flow units expressed by the length of the associated relaxation times, and used in the interpretation of flow observations in natural and synthetic rubbers, polymeric solutions, greases and metals. With the aid of the concept of "entanglement," thixotropic transformations and Ostwald's structural viscosity flow curves can be quantitatively represented.

The relation of viscous flow and diffusion is established by considering the latter as a multiple-shear process, and the nature of "holes" and "hole-formation" and its relation to the melting of solids is discussed on a statistical-mechanical basis.

The sections of the article are: I. Introduction; II. Generalized theory of viscosity; III. A generalized theory of diffusion; IV. Application; V. Nature of holes.

The article is more of physico-chemical than mathematical interest. *A. M. Freudenthal* (New York, N.Y.)

1510:

Rossow, Vernon J. On magneto-aerodynamic boundary layers. Z. Angew. Math. Phys. **9b** (1958), 519–527.

This is an extension of the same author's earlier paper [N.A.C.A. Tech. Note no. 3971 (1957)] concerning the flow near a flat plate with a transverse magnetic field. Here the electrical conductivity is assumed to vary in a specific, realistic way with the local temperature. The fluid is incompressible and has constant viscosity, and the magnetic field is approximated as uniform; i.e., induced perturbations of the field are neglected. As in the earlier paper the author considers two cases: (a) magnetic field "fixed relative to plate", and (b) magnetic field "fixed relative to fluid far from plate". (The meanings of these terms can be deduced from the author's equations; viz., the two cases represent two different boundary conditions on the electric field.) The resulting magneto-hydrodynamic effects on both skin friction and heat transfer at the wall are evaluated.

W. R. Sears (Ithaca, N.Y.)

1511:

Cowling, T. G.; and Hare, A. Two-dimensional problems of the decay of magnetic fields in magnetohydrodynamics. Quart. J. Mech. Appl. Math. **10** (1957), 385–405.

The normal modes of decay of a magnetic field in the presence of steady motions in a fluid conductor are considered in the two-dimensional case. In this case the fluid motion and the magnetic field (confined to the (x, y)-plane) can be represented respectively in terms of a stream function (ψ) and a scalar potential function A; and when these quantities vary with time like $e^{-\sigma t}$ the equations to be considered are

$$(1)\qquad \nabla^2 A=4\pi\kappa\{-\sigma A+\mathbf{k}\cdot[\operatorname{grad}\psi\times\operatorname{grad}A]\}$$

(where $\mathbf{k}$ is a unit vector parallel to Oz and κ is the electrical conductivity) inside the conductor and (2) $\nabla^2 A=0$ outside the conductor; and these equations have to be solved with suitable boundary conditions. The normal modes are determined by the characteristic values of σ. In the absence of any motions $(\psi=0)$, the normal modes (A_r) belonging to the characteristic values (σ_r) form a complete orthogonal set of functions. For small ψ, the first and the second order changes in the characteristic

values can therefore be determined by a standard perturbation method one is familiar with in quantum mechanics. The application of this theory shows that in the first approximation the motion always increases the rate of decay of the lowest mode. For the decay of the field in a slab (confined between $y=\pm a$) when motions are entirely in the x-direction, the perturbation theory shows that the field drifts in the direction of motion with a velocity V which is a weighted mean of the fluid velocity. The case of slow non-uniform rotation is also considered; and the resulting slow rotation of the fluid is determined.

For fast motions it is shown that there exists a limited class of fields little affected by motions. Such cases arise when the potential A is expressible in the form $A=A_0(\psi)+A_1$ where A_0 is a function of the stream function ψ only and $|A_1|\ll|A_0|$; and the determination of σ reduces to a classical Sturmian problem expressible in a variational form. There are, however, other types of fast motions which profoundly affect the decay. In case the velocity is in the x-direction and has the form $v=\mu y$ (μ=constant) and $\psi=-\frac{1}{2}\mu y^2$, it is shown that the basic equation can be solved in terms of Bessel functions of order $\frac{1}{3}$; and that in the case of fast motions ($\mu\to\infty$) the characteristic equation is essentially that which determines cylindrical functions of order $-\frac{2}{3}$ which vanish at two fixed points. The asymptotic form of the solutions for fast motions can be explicitly worked out; and the authors give a very complete discussion of this case. They find that the motion transfers and distorts the field while the rate of decay of the lowest mode increases roughly proportional to the $\frac{2}{3}$ power of the velocity. For velocities of intermediate magnitudes, the problem is investigated by numerical methods. The type of field lines which arise is illustrated graphically.

S. Chandrasekhar (Williams Bay, Wis.)

1512:

Backus, George. A class of self-sustaining dissipative spherical dynamos. Ann. Physics 4 (1958), 372–447.

The basic problem of self-sustaining dynamos is the following: Are there velocity fields (**u**) in a conducting incompressible fluid sphere which will maintain, for all time, a non-vanishing external magnetic field (**B**) in virtue of the equations

$$\frac{\partial \mathbf{B}}{\partial t}=\operatorname{curl}(\mathbf{u}\times\mathbf{B})-\operatorname{curl}(\operatorname{curl}\mathbf{B}) \text{ and } \operatorname{div}\mathbf{B}=0$$

inside the volume, and curl **B**=0 and div **B**=0 outside the volume? The main result of this paper is the construction of a large class of continuously differentiable fluid velocities capable of indefinitely maintaining or amplifying the dipole moment of the external magnetic field. The fluid motions shown capable of dynamo maintenance are of this sort: a toroidal shear motion symmetric about the z-axis is maintained long enough to produce from the lowest poloidal free decay mode, P_0, (symmetric about that axis) a very large energy in the lowest toroidal free decay mode, T_0 (which has the same symmetry). A period of no motion follows; and during this period everything else almost dies out leaving a field which is largely T_0. Then a velocity which has a radial component and is not axisymmetric acts for a time sufficient to regenerate P_0.

The techniques used by the author are mathematically rigorous: in particular, he shows that the fluid motions in a sphere can be regarded as bounded linear operators on the Hilbert space of magnetic fields with finite total energy. *S. Chandrasekhar* (Williams Bay, Wis.)

1513:

Roberts, P. H. Hydromagnetic disturbances in a fluid of finite conductivity. II. Astrophys. J. 126 (1957), 418–428.

This paper extends the work described in a previous paper by the author [Astrophys J. 122 (1955), 315–326; MR 17, 209], in which attention was confined to one-dimensional disturbances. Using the method of Green's functions the author obtains a general solution of the initial value problem for small-amplitude disturbances in a medium of finite conductivity. The disturbances travelling parallel to the undisturbed magnetic field are true Alfvén waves and are discussed in the author's previous paper, cited above. The disturbances travelling across the field propagate by diffusion. An approximation made by Walen [Ark. Mat. Astr. Fys. **30A**, no. 15 (1944)] is exhibited as the first term of a series expansion of the exact solution. Finally, a formal solution is given to the problem of the decay of an arbitrarily specified initial configuration. *D. Layzer* (Cambridge, Mass.)

1514:

Agostinelli, Cataldo. Su alcuni moti magneto idrodinamici ai quali è applicabile la teoria di Helmholtz sui vortici. Univ. e Politec. Torino. Rend. Sem. Mat. 16 (1956–57), 393–412.

The author shows that one set of the hydromagnetic equations may be written in the form of the equations of motion of a perfect, incompressible fluid subject to conservative forces

$$\frac{\partial \mathbf{v}}{\partial t}+\operatorname{rot}\mathbf{v}\wedge\mathbf{v}=\operatorname{grad}\Big(U-\frac{1}{\rho}\dot{p}-\frac{1}{2}v^2+\frac{\mu}{4\pi\rho}F\Big),$$

where

$$(*)\qquad F(\Phi,t)=-\int(\operatorname{grad}\Psi)^2\Phi\,d\Phi,$$

$$\nabla^2\Psi=0,\quad \operatorname{grad}\Phi\cdot\operatorname{grad}\Psi=0.$$

The vortex theorems can then be applied. If we put $(\operatorname{grad}\Psi)^2=1$, the surfaces $\Psi=$const are geodesically parallel. From (*), it follows that the surfaces are planes, and we have

$$\mathbf{H}=\Phi(x,y,t)\mathbf{k},\quad \mathbf{v}=\operatorname{grad}W\wedge\mathbf{k},$$

where **k** is a unit vector normal to the planes, W is the stream function, and **H** is the magnetic field. Thus, we have a plane motion with a perpendicular magnetic field parallel to the vortex rot $\mathbf{v}=-\nabla^2W\mathbf{k}$.

Finally, the author gives a detailed analysis of the reduced equations

$$\frac{\partial\Phi}{\partial t}+\operatorname{grad}\Phi\wedge\operatorname{grad}W\cdot\mathbf{k}-\frac{c^2}{4\pi\mu\sigma}\nabla^2\Phi=0,$$

$$\operatorname{grad}\frac{\partial W}{\partial t}\wedge\mathbf{k}-\nabla^2W\operatorname{grad}W=-\operatorname{grad}\Pi,$$

obtained from the full set of hydromagnetic equations by means of his successive hypotheses.

C. D. Calsoyas (Livermore, Calif.)

1515:

Carini, Giovanni. Sulle equazioni della magneto-idrodinamica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 21 (1956), 436–441.

Una modificazione delle equazioni delle onde di Alfvén in magneto-idrodinamica è qui proposta, applicando un'osservazione del recensore, secondo la quale una migliore approssimazione si consegue nell'indagine delle onde di Alfvén se alle equazioni di Eulero dell'idrodinamica si

associano le equazioni di Minkowski dei conduttori in moto, anzichè quelle di Maxwell, come si fa ordinariamente. *G. Lampariello* (Roma)

1516:

Bhatnagar, P. L. The equilibrium of a self-gravitating incompressible fluid sphere with magnetic field. J. Indian Inst. Sci. Sect. A. **40** (1958), 50–73.

If a sphere of infinitely conducting material is to be in equilibrium under the action of its pressure-gradient force, its gravitational self-attraction and an imposed magnetic field **H**, it is easy to show that

$$\nabla\times(\nabla\times\mathbf{H})\times\mathbf{H}=\mathbf{0}.$$

Assuming that **H** possesses cylindrical symmetry, a general solution of this equation is found and the particular solutions obtained by other authors are obtained as special cases. A toroidal magnetic field is dealt with in some detail, the distortion of the sphere being calculated and boundary conditions applied. The configuration is a prolate spheroid. The change of the total magnetic energy between the spherical and prolate spheroid configurations is evaluated. An equilibrium configuration when a toroidal field and a poloidal field are both present is also worked out in detail.

G. C. McVittie (Urbana, Ill.)

1517:

Goto, Ken-iti. Relativistic magnetohydrodynamics. Progr. Theoret. Phys. **20** (1958), 1–14.

Starting with the Boltzmann equation in a form appropriate to special relativity, the energy tensor of a fluid is defined and the equations governing the motion of the fluid and the conservation of mass and of energy are established. There are some curious features in this development: the metrical tensor is defined by its contravariant, instead of by the usual covariant, form; the interval is introduced by considering the 4-vector velocity of a particle and is denoted by $cd\tau$ instead of the usual ds; it is said that a symmetric tensor $s^{\lambda\mu}$ can be orthogonal to a vector V_λ as if the tensor possessed a direction. What is apparently meant is that the vector $V_\lambda s^{\lambda\mu}$ is orthogonal to V_λ.

The treatment of the material density of the flow, of the energy density and of the internal energy is essentially the same as that given by Taub [Phys. Rev. (2) **74** (1948), 328–334; MR **10**, 72] except that Goto also tries to express the internal energy in terms of the departures of the velocities of the particles from the mean flow velocity. His definition of the internal energy follows his equation (3.24). If the internal energy so defined is small, the classical theory should apply; but his equations (3.13), (3.16), (3.24), (6.3) and (6.4) then lead to the following definition for the entropy s for a gas of temperature T:

$$TdS=\tfrac{1}{2}de+pd(1/\rho),$$

e being the internal energy per unit mass. It is well-known that the factor $\frac{1}{2}$ should not be present.

The fully ionized and the partially ionized fluid are discussed in detail as applications of the general theory developed in the first part of the paper. The proofs of the circulation and vortex theorems for isentropic flows of simple perfect fluids are given and the conditions at a shock front are established for one-dimensional flow in a uniform magnetic field. *G. C. McVittie* (Urbana, Ill.)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 1373.

1518:

Morgan, Samuel P. General solution of the Luneberg lens problem. J. Appl. Phys. **29** (1958), 1358–1368.

The author solves the problem of constructing a Luneberg lens; i.e., the concentric inhomogeneous medium imaging one concentric sphere sharply on another. He shows that previous solutions by Luneberg, Brown, and Gutman are special cases of his general solution. He gives numerical samples and a table to compute the index of refraction for such a lens.

M. Herzberger (Rochester, N.Y.)

1519:

Linfoot, E. H. Optical image evaluation from the standpoint of communication theory. Physica **24** (1958), 476–494.

This thought-provoking but difficult paper attempts to transfer the concepts developed in communication theory to the evaluation of the optical and photographic image and its information content in the presence of optical "noise". In this connection, the author studies the different attempts to assess the quality of an optical system, especially with respect to information details which could be reconstructed by an arithmetical decoding process. The author suggests a method which can give a quantitative approach for the best matching of optical system and emulsions. The paper suggests, among other things, that any evaluation of an optical system should include an investigation of focus shifts, since for detail of varying size, various positions of the focus are indicated.

M. Herzberger (Rochester, N.Y.)

1520:

Itô, Hirosi. Asymptotic behaviour of the response function of an optical system with the primary spherical aberration. J. Phys. Soc. Japan **13** (1958), 639–649.

Calculation of the sine-wave response for the out-of-focus image of an axis point in the presence of spherical aberration. The response function is calculated and evaluated with the help of a Fresnel integral, for which a first and second approximation are computed. The result is compared with experimental data.

M. Herzberger (Rochester, N.Y.)

1521:

Joshi, G. H. The electromagnetic interaction between two crossing electron streams (with Lorentz term). Chalmers Tekn. Högsk. Handl. no. **197** (1958), 10 pp.

The paper deals with the electromagnetic interaction between two crossing electron streams. The treatment takes into account the Lorentz term in the equation of motion which results in comparatively difficult analytical expressions. On the basis of numerical computations it is concluded that the Lorentz term may be omitted at low beam velocities common in microwave tubes but not at higher velocities common in accelerators. The relativistic correction needed in the latter case is also considered.

J. E. Rosenthal (Passaic, N.J.)

1522:

Wilhelmsson, Hans. The interaction between an obliquely incident plane electromagnetic wave and an electron beam. II. Chalmers Tekn. Högsk. Handl. no. **198** (1958), 32 pp.

This paper, an extension of earlier work [same Handl., no. **155** (1954)], deals with the problem in the absence of a static magnetic field. The treatment is relativistic and takes into account Lorentz forces. The *dc* parameters are

assumed not to vary in the direction of the axis. Two coupled differentials equations for the field components are obtained and the conditions for their separation into pure *TM* and *TE* modes are discussed. It is found that separation is possible even for oblique incidence if the electron density is constant over the beam cross section. Solutions are obtained for cases when the magnetic and electric field vectors, respectively, are perpendicular to the electron beam. Superposition of these solutions gives the complete solution for the general case of an arbitrarily polarized wave. *J. E. Rosenthal* (Passaic, N.J.)

1523:

Kockel, Bernhard. Die Sommerfeldsche Bodenwelle. Ann. Physik (7) **1** (1958), 145–156.

This is a critical discussion of the old paper by Sommerfeld [Ann. Physik (4) **28** (1909), 665–736].

A. E. Heins (Pittsburgh, Pa.)

1524:

Marziani, Marziano. Sulla integrazione delle equazioni di Maxwell. Rend. Sem. Mat. Univ. Padova **27** (1957), 80–89.

Il metodo della trasformazione di Laplace viene applicato per l'integrazione delle equazioni di Maxwell nel caso dei mezzi cristallini biassici dielettrici. L'applicazione del metodo conduce ad un'equazione vettoriale simbolica che permette di stabilire la connessione tra le grandezze del campo elettromagnetico e i loro valori al contorno.

Il vantaggio del procedimento in confronto alla trattazione di problemi concernenti dielettrici isotropi e mezzi cristallini uniassici, dovuta ad altri Autori, sta nella possibilità di evitare il teorema di reciprocità.

G. Lampariello (Roma)

1525:

Rosman, Hugo; et Cramariuc, Radu. Sur la conservation de l'énergie dans les milieux à viscosité électromagnétique. Bul. Inst. Politehn. Iaşi (N.S.) **2**(6) (1956), 49–52. (Romanian. Russian and French summaries)

1526:

Schubert, G. U.; und Schmauch, H. Der supraleitende elliptische Zylinder im transversalen Magnetfeld. Z. Physik **151** (1958), 396–407.

This mathematical paper considers the field and current distribution inside an infinitely long, superconducting, elliptical cylinder which is placed in a transverse magnetic field. The London theory is used for the calculation, which is carried out in elliptical cylindric coordinates. It is assumed that the magnetic field is transverse before the Meissner effect sets in. Special expansions are given for the case when the eccentricity of the ellipse is small. Beside the formulae a graph is also given to show the dependence on the eccentricity.

M. J. Moravcsik (Livermore, Calif.)

1527:

Lugaresi, Erminia. Su alcuni teoremi di reciprocità dell'elettromagnetismo. Boll. Un. Mat. Ital. (3) **12** (1957), 443–445.

The author demonstrates that some of the interesting reciprocity theorems put forth by Nicolau [Acad. R.P. Romine, Bul. Şti. Şect. Şti. Mat. Fiz. **4** (1952), 739–749; MR **15**, 585] are identical with Lorentz's theorems [cf. Riemann and Weber, Differentialgleichungen der Physik, Braunschweig, 1927, vol. II, p. 575] for harmonically varying electro-magnetic fields. *K. Bhagwandin* (Oslo)

1528:

***Аэрилянт, П. А.; и Белкина, М. Г. Численные результаты теории дифракции радиоволн вокруг земной поверхности. [Azrilyant, P. A.; and Belkina, M. G. Numerical results of the theory of diffraction of radiowaves around the earth's surface.] 2nd. ed.** Izdat. Soviet. Radio, Moscow, 1957. 45 pp. (44 inserts) 8.80 rubles.

This book, containing numerical computations of the attenuation factor in the propagation of electromagnetic waves over the earth, is addressed primarily to radio engineers. The calculations are made from the residue-series expansions of the integrals of V. A. Fok for the attenuation factor [Ž. Eksper. Teoret. Fiz. **19** (1949), 916–929; MR **11**, 563]. The results are presented in the form of extensive tables and graphs for various conditions of propagation. *R. N. Goss* (San Diego, Calif.)

1529:

Voelker, Dietrich. Application of the Laplace transformation to the diffraction by irregular nets. Rev. Un. Mat. Argentina **18** (1956), 3–15. (Spanish)

The author applies the Laplace transform to solve one-dimensional diffraction problems (light) by irregular nets.

Kirchhoff's integral representation is the central theme. The inversion is obtained in terms of ϑ-type functions. The analysis is elementary. References to related treatments are lacking. *K. Bhagwandin* (Oslo)

1530:

Zeuli, Tino. Perturbazione di onde elettromagnetiche guidate di tipo TEM entro un cavo cilindrico circolare, dovuta a piccole eterogeneità del dielettrico. Boll. Un. Mat. Ital. (3) **12** (1957), 22–33.

L'A. ricerca le modificazioni che subiscono le grandezze del campo di onde elettromagnetiche, se si suppone che il mezzo dielettrico non sia perfettamente omogeneo, ma si ammette invece che la variazione della costante dielettrica dallo stato di omogeneità sia una funzione del posto, entro la cavità che guida le onde, soddisfacente a determinate condizioni. *G. Lampariello* (Roma)

1531:

Napolitano, Luigi G. Su di un'equazione incontratasi nello studio della interazione di due correnti. Ricerca, Napoli (2) **8** (1957), Luglio-Dicembre, 39–59.

1532:

***Rogers, F. E. The theory of networks in electrical communication and other fields.** D. Van Nostrand Co., Inc., Princeton, N. J.-Toronto-London-New York, 1958. 560 pp. $11.50.

This book covers the general field of linear communication and network theory. After a brief introduction, fundamental concepts of linear network theory, as based on Kirchhoff's laws, are discussed. Transient and steady state conditions are illustrated by means of elementary circuits, and the use of Fourier integrals for the analysis of non-sinusoidal E.M.F.'s is briefly shown (chapters 1, 2). The general equations of lumped systems and their solution by determinants are given (chapter 3) and some of the most important general network theorems are discussed (chapter 4). Problems of structure and network equivalence are introduced (chapters 5, 6). The behavior of particular systems is discussed: two-terminal networks (chapter 7) and uniform lines and cables and their transmission characteristics (chapter 8). Problems referring to four-terminal networks — transmission characteristics, impedance matching, principles of filter networks — are

treated (chapters 9, 10, 11). Finally the principal measurement methods are discussed (chapter 12). The book confines itself mainly to steady state problems, therefore integral transform methods are omitted. The exposition is systematical, the mathematical treatment didactical, clear, and rather elementary, and intermediate results are included in full detail. Many fully worked-out examples are included. Frequent references to the underlying physical problems and the treatment of measurements help to avoid the impression that network theory is but a branch of mathematics. A particularly useful feature is the systematic treatment of the derivation, significance, and application of important network theorems (chapter 4). The book is to be recommended as a general text in engineering courses on communication and for self-study.

B. Gross (Rio de Janeiro)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See 1485, 1486, 1487.

QUANTUM MECHANICS

See also 1002, 1065, 1448, 1449, 1563.

1533:

Baker, George A., Jr. Formulation of quantum mechanics based on the quasi-probability distribution induced on phase space. Phys. Rev. (2) **109** (1958), 2198–2206.

Wigner [Phys. Rev. (2) **40** (1932), 749–759] introduced in quantum mechanics a quasi-probability function, investigated later by Groenewold [Physica **12** (1946), 405–460; MR **8**, 301], Moyal [Proc. Cambridge Philos. Soc. **45** (1949), 99–124; MR **10**, 582], and Takabayasi [Progress Theoret. Physics **11** (1954), 341–373; MR **16**, 319]. The author formulates quantum mechanics in terms of such a function $f(q_k, p_k, t)$ subject to normalization, $\int f dq dp = 1$; square integrability, $\int |f|^2 dq dp$ finite; quantum condition, $f = h^n(f, f)$; and dynamical condition, $\partial f/\partial t = -h^{-1}[f, H]$. He shows that this formulation is equivalent to the standard one, that f is analogous to the Dirac-von Neumann density matrix and that the solution of the correspondence problem between classical functions and quantum operators of conjugate position and momentum is equivalent to the solution of the eigenvalue problem; and he gives a constructive method for solving the last problem.

M. Loève (Berkeley, Calif.)

1534:

Łopuszański, J. On the modification of Feynman's "Integral-over-all-paths" method. Nuovo Cimento (10) **7** (1958), supplemento, 61–76.

The general solution of the diffusion equation

$$\psi(j, j_0; n) = \sum_{j'=1}^{l} z(j|j')\psi(j', j_0; n-1)$$

can be written $\sum_{x_1=1}^{l} \cdots \sum_{x_{n-1}=1}^{l} \prod_{r=1}^{n} z(x_r|x_{r-1})$ if there are a finite set of states. By treating these equations as a finite linear system, the author obtains a representation of the solution as a contour integral, from which an approximation for large n is derived. Using central limit type theorems, the appropriate form of this approximation for n a continuous variable is given. It is claimed that this method may be useful in evaluating Feynman's "integral-over-all-paths".

C. A. Hurst (Adelaide)

1535:

Bogoliubow, N. N.; und Parasiuk, O. S. Über die Multiplikation der Kausalfunktionen in der Quantentheorie der Felder. Acta Math. **97** (1957), 227–266.

Une "fonction causale" est une distribution Δ^c en $x \in R^4$ dont l'image de Fourier est de la forme $\lim_{\varepsilon \to 0} P(k)/(m^2 - k^2 - i\varepsilon)$, $k \in R^4$, k^2 valeur en k de la forme quadratique fondamentale, m masse, P polynome Lorentz-invariant. Les distributions Δ^c sont singulières sur le cône d'ondes. Le but de cet article est de définir des produits multiplicatifs $P = \prod_l \Delta_l^c(x_r - x_s)$, où les x_r, $r = 1, 2, \cdots, n$, sont des points génériques de R^4, et les $l = 1, 2, \cdots, L$, des segments joignant deux de ces points (x_r, x_s); ce produit, si on peut le définir, est une distribution en $x_1, x_2, \cdots, x_n$.

L'auteur utilise d'abord la régularisation de Pauli-Villars. Chaque distribution Δ_l^c est remplacée par sa régularisée $\Delta_{l,M}^c$, obtenue en ajoutant à Δ_l^c une combinaison linéaire convenable de Δ_{l,M_j}^c correspondant à des masses M_j; on sait qu'alors $\Delta_{l,M}^c$ est une fonction partout continue, et que, si les M_j tendent vers l'infini de manière convenable, $\Delta_{l,M}^c$ converge (au sens des distributions) vers Δ_l^c. On peut former le produit de régularisées, $P_M = \prod_l \Delta_{l,M}^c(x_r - x_s)$, produit de fonctions continues. Lorsque les M_j tendent vers l'infini comme indiqué ci-dessus, le produit P_M ne converge pas vers une limite, mais $P_M(\varphi)$ converge vers une limite $P(\varphi)$ si φ est nulle ainsi qu'un certain nombre de ses dérivées lorsque l'un au moins des couples (x_r, x_s) est diagonal $(x_r = x_s)$, avec $(r, s) = l = 1, 2, \cdots, L$.

Il reste ensuite à définir $P(\varphi)$ pour toute $\varphi \in \mathcal{D}_{x_1, x_2, \cdots, x_n}$. L'auteur y parvient par l'"opération R" ou opération de soustraction, trop compliquée pour être décrite ici. Il utilise les diagrammes combinatoires formés par les points x_r, $r = 1, 2, \cdots, n$, et les segments (x_r, x_s) donnés qui les joignent. Le produit P ainsi défini est tel que la matrice S satisfasse aux axiomes de causalité, d'unitarité, et d'invariance-Lorentz. C'est l'étude détaillée de cette opération R et de ses propriétés qui est la partie la plus importante de l'article, tant au point de vue du formalisme mathématique que des conséquences physiques.

L. Schwartz (Paris)

1536:

Laurent, B. E. On covariant quantization with application to the scattering of gravitating Dirac particles. Nuovo Cimento (10) **4** (1956), 1445–1460.

In this paper the author reproduces the work of the reviewer on the quantization of the gravitational field Proc. Phys. Soc. Sect. A. **65** (1952), 161–169, 608–619; MR **13**, 804; **14**, 417] with the difference that instead of the standard field theory he has made use of some intuitive arguments.

S. N. Gupta (Detroit, Mich.)

1537:

Dyson, Freeman J. Integral representations of causal commutators. Phys. Rev. (2) **110** (1958), 1460–1464.

The problem discussed in this paper is the following. Find a representation of the most general function $f(q)$ depending on a four vector q and such that (i) its Fourier transform $\bar{f}(x) = \int d_4 q \exp[iqx] f(q)$ is zero for space-like vectors x and (ii) the function $f(q)$ itself vanishes inside a given region R in q-space bounded by two space-like sur-

faces σ_1 and σ_2. This problem is solved and the answer given in the following form.

$$f(q)=\int d_4u \int ds\, \varepsilon(q_0-u_0)\delta[(q-u)^2-s]\psi(u, s),$$

where the weight function $\psi(u, s)$ vanishes outside a certain region S but is otherwise arbitrary. The region S can be specified when R is known but the expressions are somewhat involved and will not be reproduced here.

The proof of the formula is done in two steps. First is shown that any function $f(q)$ such that $\hat{f}(x)$ vanishes for space-like x can be extended to a solution of the wave equation with zero mass in a q-space with two extra space-like dimensions. {The reason why exactly two dimensions were chosen is not clear. The argument seems to work as well with an arbitrary number (also one!) of extra dimensions. The reviewer is indebted to L. Gårding for this remark.} The function is then characterized with the aid of its Cauchy data on a general space-like surface Σ. The difficult part of the proof is then to show that these Cauchy data have to vanish for every point of Σ such that it can be connected with a light signal (in the six-dimensional space) to any point in R. Once this is established the representation given above follows.

G. Källén (Lund)

1538a:

Lüders, Gerhart; and Zumino, Bruno. Connection between spin and statistics. Phys. Rev. (2) **110** (1958), 1450–1453.

1538b:

Burgoyne, N. On the connection of spin with statistics. Nuovo Cimento (10) **8** (1958), 607-609.

These two papers independently reconsider Pauli's celebrated theorem on the connection of spin with statistics, [See W. Pauli in "Niels Bohr and the development of physics", McGraw-Hill, New York and Pergamon Press, London, 1955, for references] and prove it under weaker hypotheses. In technique, the proof is closely related to R. Jost's treatment of the PCT theorem [R. Jost, Helv. Phys. Acta **30** (1957), 409–416; MR **19**, 712]. The first paper confines itself to spin-zero and spin-one-half fields and in the non-hermitean case introduces an hypothesis of gauge invariance. The second paper treats the hermitean and non-hermitean cases together for arbitrary spin omitting the unnecessary hypothesis of gauge invariance.

Here is the statement of the theorem and the main idea of the proof following the latter paper: In a relativistic field theory with a vacuum state, Ψ_0 (by definition the state of lowest energy), and a positive definite scalar product $(\ ,\ )$ in the Hilbert space of states, no non-trivial field can have the wrong connection of spin with statistics. (The wrong connection of spin with statistics means that if ϕ transforms according to an irreducible representation of the homogeneous Lorentz group labeled by $[j_1, j_2]$, where j_1, j_2 are integers or half odd integers, $\phi_\alpha(x)\phi_\beta^*(y)+(-1)^{2(j_1+j_2)}\phi_\beta^*(y)\phi_\alpha(x)=0$ for $(x-y)$ space-like) Proof: Consider the distribution $F_{\alpha\beta}^{\phi\phi^*}(x_1-x_2)=(\Psi_0, \phi_\alpha(x_1)\phi_\beta^*(x_2)\Psi_0)$. A straightforward argument using only the properties of the representations $[j_1, j_2]$ shows that $F_{\alpha\beta}^{\phi\phi^*}(\xi)=(-1)^{2(j_1+j_2)}F_{\alpha\beta}^{\phi\phi^*}(-\xi)$ for ξ space-like. The hypothesis of the wrong connection of spin with statistics implies $F_{\alpha\beta}^{\phi\phi^*}(\xi)+(-1)^{2(j_1+j_2)}F_{\beta\alpha}^{\phi^*\phi}(-\xi)=0$ for ξ space-like. Together these imply

(1) $$F_{\alpha\beta}^{\phi\phi^*}(\xi)+F_{\beta\alpha}^{\phi^*\phi}(\xi)=0, \text{ for } \xi \text{ space-like.}$$

Because $F_{\alpha\beta}^{\phi\phi^*}(\xi)$ is (real) analytic for ξ space-like and can be continued to real all $\xi-i\eta$ with η in the future light cone, one gets (1) for all ξ as a relation between distributions by passing to the boundary of the analyticity domain, $\eta\to 0$. Now if one defines $\phi(f)=\int dx \sum_\alpha f_\alpha(x)\phi_\alpha(x)$, $\phi^*(f)=\phi(\bar{f})^*$ and $\bar{f}_\alpha(x)=\bar{f}_\alpha(-x)$, one gets

$$\|\phi(f)\Psi_0\|^2+\|\phi^*(f)\Psi_0\|^2=\int dxdy \sum_{\alpha\beta} \bar{f}_\alpha(x)$$

$$[F_{\alpha\beta}^{\phi\phi^*}(x-y)+F_{\beta\alpha}^{\phi^*\phi}(x-y)]f_\beta(y)=0.$$

Thus,

(2) $$\phi(f)\Psi_0=0=\phi^*(f)\Psi_0,$$

and $\phi(f)$ is trivial.

(It is assumption V of Lüders and Zumino that fields satisfying (2) are excluded from consideration. Burgoyne shows that $\phi(f)=0$, under the additional assumption that all fields of the theory commute or anti-commute.) The proof has the great advantage over other treatments that it deals with interacting as well as non-interacting fields without assumptions about how the Hamiltonian or Lagrangian is expressed in terms of fields. For an independent discussion which is also based on the ideas of Jost but involves such irrelevant assumptions see J. Schwinger, Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 223–228 [MR **18**, 1232]. *A. S. Wightman* (Princeton, N.J.)

1539:

*Schwinger, Julian (Editor). **Selected papers on quantum electrodynamics.** Dover Publications, Inc., New York, 1958. xvii+424 pp. $2.45.

A collection of articles by Dirac, Fermi, Fock, Podolsky, Jordan, Wigner, Heisenberg, Weisskopf, Bloch, Nordsieck, Foley, Kusch, Lamb, Retherford, Bethe, Schwinger, Oppenheimer, Tomonaga, Pauli, Villars, Feynman, Dyson, Karplus, Klein, Källen and Kroll. They date from 1927 to 1953.

1540:

Tanikawa, Yasutaka; and Watanabe, Stosi. Chirality of tensors and parity-nonconserving interactions. Phys. Rev. (2) **110** (1958), 289–290.

The authors investigate the most general type of transformations, which keep the interaction Lagrangian density invariant for the electromagnetic interaction, the pion-nucleon interaction, and the parity-nonconserving neutrino interactions.

S. N. Gupta (Detroit, Mich.)

1541:

Sunakawa, Sigenobu. Quantum electrodynamics with the indefinite metric (non-Lorentz-invariance of the Gupta formalism). Progr. Theoret. Phys. **19** (1958), 221–237.

The author points out several advantages of the reviewer's formalism of quantum electrodynamics with the indefinite metric [S. N. Gupta, Proc. Phys. Soc. London Ser. A **63** (1950), 681–691; MR **12**, 67], but claims that it is not Lorentz covariant. This claim of the author is quite unjustified, because the Lorentz covariance of quantum electrodynamics with the indefinite metric has been clearly established by Belinfante [F. J. Belinfante, Phys. Rev. (2) **96** (1954), 780–787; MR **16**, 432] and several other authors. The error in the present paper arises from the fact that in the formalism of indefinite metric two sets of quantities are introduced, which are called the Hermitian conjugates and the adjoints, and the author has confused the two.

In a recent paper [S. N. Gupta, Canad. J. Phys. **35**

(1957), 961–968; MR **19**, 711], the reviewer has given a simplified formalism of quantum mechanics with the indefinite metric. In this formalism only one set of quantities is introduced, and therefore the proof of Lorentz covariance becomes much simpler.

S. N. Gupta (Detroit, Mich.)

1542:

Pócsik, György. Theorems of conservation in the bilocal field theory of Rayski. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **7** (1957), 163–178. (Hungarian)

Bekanntermeise rührt der Gedanke einer bilokalen (oder nichtlokalen) Feldtheorie, nach der die Feldgrössen Funktionen von nicht einer sondern zwei Scharen von vier Parametern(Koordinaten) sind, von H. Yukawa her. [Vgl., z.B., Proc. Internat. Conference Theoret. Phys., Kyota and Tokyo, 1953, Science Council of Japan, Tokyo, 1954, 2–6; MR **16**, 656]. Die Berechnungen des Verfassers schliessen sich an die von J. Rayski [Acta Phys. Polonica **14** (1955), 337–364; **15** (1956), 123–141; MR **19**, 366; **18**, 260] modifizierte Theorie an. Mit Hilfe der Invarianzeigenschaften der Lagrange-Dichte (deren Integral über den ganzen Raum das wellenmechanische Analogon der klassischen Lagrange-Funktion ist), werden die Erhaltungssätze in bilokalen Wellenfeldern streng formuliert und dabei wird gezeigt, wie die Bilokalität die erwähnten Sätze beeinflusst und wie man die anschaulich interpretieren kann. Als Beispiele werden das bilokale Boson-Feld und das bilokal verallgemeinerte Diracsche Spinor-Feld besprochen.

T. Neugebauer (Budapest)

1543:

Young, R. C.; Biedenharn, L. C.; and Feenberg, E. Continued fraction approximants to the Brillouin-Wigner perturbation series. Phys. Rev. (2) **106** (1957), 1151–1155.

It is shown that the refined formulae for the energy proposed by Goldhammer and Feenberg [Phys. Rev. (2) **101** (1956), 1233–1234] are to be identified with alternate approximants of a continued fraction of which the original Brillouin-Wigner perturbation series for the energy is the so called "formally equivalent linear series". This identification is essential for the question of convergence. Further, the authors prove the invariance of the alternate approximants under a particular class of transformation of the zeroth-order Hamiltonian.

D. Rivier (Lausanne)

1544:

Gourdin, M.; et Martin, A. Interaction non locale séparable et matrice de collision. Nuovo Cimento (10) **6** (1957), 757–779.

The authors consider the problem of determining the interaction between two particles in the S state from the relation between the experimental phase shift and the energy. They assume a separable non-local interaction of the form

$$\varepsilon u(r)u(r').$$

Writing

$$pv(p)=4\pi\int_0^\infty ru(r)\sin pr\,dr,$$

$$f(p)=\frac{\varepsilon M}{2\pi^2}p^2v^2(p),$$

M being the reduced mass of the system, they show that the phase shift δ is given by

$$-\frac{2}{\pi}k\tan\delta=g(k),$$

where $f(k)$ and $g(k)$ are related by the singular linear integral equation

$$f(k)=g(k)\left\{1+P\int_0^\infty\frac{f(p)}{p^2-k^2}\,dp\right\}.$$

They show that:

(i) if $g(+0)<0$, $g(+\infty)>0$, the equation has no physically admissible solution for $f(k)$; (ii) if $g(+0)\cdot g(+\infty)>0$, the solution, if it exists, is unique; (iii) if $g(+0)>0$, $g(+\infty)<0$, there is at most one linearly independent solution of the associated homogeneous equation. Solutions of the homogeneous equation correspond to bound states of the physical system.

Explicit solutions are given for the cases

$$g(k)=\frac{ck^2}{k^2+\alpha^2},\quad g(k)=\frac{ck^2}{k^2-\gamma^2},$$

and the case where $g(k)$ is a more general rational function of k is analysed in some detail.

F. Smithies (Cambridge, England)

1545:

Froese, Charlotte. The limiting behaviour of atomic wave functions for large atomic number. II. Proc. Roy. Soc. London. Ser. A. **244** (1958), 390–397.

The study of the limiting behaviour of the atomic self-consistent-field wave functions for large atomic number N in a previous paper [C. Froese, Proc. Roy. Soc. London Ser. A **239** (1957), 311–319; MR **19**, 811], have been extended to the $N^{-\frac{2}{3}}$-term in the asymptotic expansion. Explicit so lutions have been evaluated for the configurations having the outer groups $(2p)^6$, $(3s)^2$, $(3p)^6$, $(3d)^5$, and $(3d)^{10}$. The screening $\sigma(nl)$ has been investigated as a function of the mean radius $\bar{r}$; the results for $(2p)^6$ indicate that this quantity is an almost linear function of $\bar{r}$ for all values of N, whereas a large deviation from linearity was discovered for $\sigma(3s)$ and $\sigma(3p)$ of the configuration $(3d)^{10}$. The results may be used for interpolating or extrapolating atomic wave functions to a higher degree of accuracy.

P.-O. Löwdin (Uppsala)

1546:

Khalatnikov, I. M.; and Abrikosov, A. A. Dispersion of sound in a Fermi liquid. Soviet Physics. JETP **6** (1958), 84–89.

The starting point of the calculation is the kinetic equation for the excitations in a Fermi liquid derived by Landau. After some simplifications Landau obtained from this equation the expression for the occurrence of undamped oscillations at absolute zero (zero sound). At temperatures different from zero the ordinary sound propagates in the Fermi liquid. Because of the unknown forms of certain functions, some far-reaching simplifications were proposed by the authors, which enabled them to obtain the complex sound velocity in a Fermi liquid. Specific calculations are performed for a liquid He^3 and for two limiting cases of frequencies: low and high.

M. Z. v. Krzywoblocki (Urbana, Ill.)

1547:

Silin, V. P. Oscillations of a Fermi-liquid in a magnetic field. Soviet Physics. JETP **6** (1958), 945–950.

Landau [Soviet Phys. JETP **5** (1957), 101–108; MR **19**, 786] investigated oscillations of a Fermi-liquid and demonstrated the possibility of the propagation of neutral sound waves in liquid He^3 at low temperatures but not of spin waves. Silin investigates spin oscillations in a Fermi liquid placed in a magnetic field whose presence was neglected by Landau. Starting from the new kinetic equations in the presence of a magnetic field, Silin derives

an equation for the characteristic frequencies of these oscillations and studies the spin waves. As a consequence, he demonstrates the possibility of the propagation of spin waves in liquid He^3 in the magnetic field.

M. Z. v. Krzywoblocki (Urbana, Ill.)

1548:

Žislin, G. M. On the existence of characteristic functions for Schrödinger's equation. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 931–934. (Russian)

Let

$$H=\sum_{i=1}^{n} a_i\left(\frac{\partial^2}{\partial x_i^2}+\frac{\partial^2}{\partial y_i^2}+\frac{\partial^2}{\partial z_i^2}\right)-\sum_{i=1}^{n} b_i(x_i^2+y_i^2+z_i^2)^{-\frac{1}{2}}+\sum_{\substack{i=1\\ j>1}}^{n} c_{ij}[(x_i-x_j)^2-(y_i-y_j)^2-(z_i-z_j)^2].$$

It is known [T. Kato, Trans. Amer. Math. Soc. **70** (1951), 212–218; MR **12**, 781] that for $n=2$ and a_i, b_i, c_{ij} corresponding to a helium atom, H has a discrete set of eigenvalues (and the corresponding eigenfunctions). The author proves that for any n, and positive a_i, b_i, c_{ij} satisfying $b_i>\sum_{j=1}^{n} c_{ij}$ $(j\neq i,\ i=1, 2, \cdots, n)$, H has a countable number of eigenvalues, each one of finite multiplicity, and that the corresponding eigenfunctions are analytic at all points for which the operator H is defined. This proves the existence of eigenvalues and eigenfunctions for the Schroedinger energy operator with Coulomb potential and an infinite mass of a nucleus.

To prove this, the author defines a space W of all measurable functions defined on $R=\{(x_1, y_1, z_1, \cdots, x_n, y_n, z_n)\}$ such that $\|\psi\|=\int_R (|\psi|^2+|\text{grad } \psi|^2)d\Omega<\infty$ where $\|\psi\|=(\psi, \psi)^{\frac{1}{2}}$ and $(\psi, \phi)=\int_R \psi\phi d\Omega$.

Let $Q_0=\{\psi|\psi\in W,\ \|\psi\|=1\}$, $Q_p=\{\psi|\psi\in Q_0,\ (\psi, \phi_k)=0, k=1, \cdots, p-1\}$ (ϕ_k defined below). It is shown that there exist functions ϕ_p $(p=0, 1, 2, \cdots)$ for which $\lambda_p=(H\phi_p, \phi_p)=\inf[(H\psi, \psi), \psi\in Q_p]$ and $\bar{H}\phi_p=\lambda_p\phi_p$ where $\bar{H}$ is selfadjoint closure of H. Then by K. Friedrichs' theorem [Amer. J. Math. **61** (1939), 523–544] it follows that ϕ_p satisfy the differential equations $H\phi_p=\lambda_p\phi_p$.

C. Masaitis (Havre de Grace, Md.)

1549:

Viswanathan, K. S. Anharmonicity of vibration in molecules. Proc. Indian Acad. Sci. Sect. A. **47** (1958), 85–97.

The self-consistent field method of Hartree is used to study the interaction of normal modes of molecular vibration caused by anharmonicity in the potential function. To the first approximation antisymmetric normal modes are unaffected by the anharmonicity. The wave functions and eigenvalues of the normal modes are evaluated to the second order. The splitting of degenerate modes is investigated using group theoretical methods.

A. C. Hurley (Melbourne)

1550:

Altmann, S. L. The cellular method for a close-packed hexagonal lattice. Proc. Roy. Soc. London. Ser. A. **244** (1958), 144–152.

"The general theory of the irreducible representations of a space group with two atoms per unit cell is discussed. A particular application of it is made for the group of $\mathbf{k}=0$ for the close-packed hexagonal lattice. This leads to the determination of the spherical harmonics with the symmetry of this group.

"A technique is described to determine the boundary and continuity conditions on the surface of the Wigner-Seitz polyhedron. It is pointed out that these vary for different types of points on this surface and complete tables for them are given." (Author's summary.)

A. C. Hurley (Melbourne)

1551:

Altmann, S. L. The cellular method for a close-packed hexagonal lattice. II. The computations: a program for a digital computer and an application to zirconium metal. Proc. Roy. Soc. London. Ser. A. **244** (1958), 153–165.

"The techniques for fitting boundary conditions in the cellular method are discussed. A difficulty that has often arisen in this respect is illustrated and its origin explained. A new method is introduced that involves the development of a technique for dealing with a least squares problem for a system of homogeneous equations subject to a subsidiary condition.

"A program for a digital computer is described which carries out the cellular computation in an entirely automatic fashion. As a part of this program, a simplified method to integrate Schrödinger's radial equation in floating-point form is described.

"The energy eigenvalues for all the 12 irreducible representations for $\mathbf{k}=0$ for zirconium metal are given." (Author's summary.)

A. C. Hurley (Melbourne)

1552:

Brodersen, Svend; and Langseth, A. A complete rule for the vibrational frequencies of certain isotopic molecules. Mat.-Fys. Skr. Danske Vid. Selsk. **1** (1958), no. 5, 21 pp.

"The existence of an isotopic rule in addition to the product rule and the sum rule is demonstrated. This rule allows the calculation of all the frequencies of certain partly isotopically substituted compounds exclusively from the frequencies of the corresponding full-symmetry compounds.

"An equation between the secular determinants of the classes in question is a convenient expression for the rule. A detailed description is given of the procedure for setting up this equation. The conditions for the validity of the rule are discussed, the result being roughly that the full-symmetry compounds should be either linear and symmetrical or plane, but of higher symmetry than C_s. Some of the rules are strictly valid within the harmonic approximation (and in fact better); others are based on the further approximation of separating low and high frequencies. In a special chapter the rules are formulated for molecules of some of the more important point groups.

"The rule is used to calculate the frequencies of all the partly deuterated species of acetylene and ethylene. The agreement with observed values is at least as good as for calculations based on a harmonic potential function." (Authors' synopsis.)

A. C. Hurley (Melbourne)

1553:

Artmann, Kurt. Die Bedeutung der Hartreeschen Atomeigenfunktionen für die Quantenchemie. Z. Physik **150** (1958), 573–589.

The influence of the inter-atomic error on molecular energy calculations is investigated. It is shown that this error, which vanishes identically if exact atomic wave functions are employed, is very small for an orbital calculation in which Hartree functions are used for the basic atomic orbitals, but that if Slater functions are used the error may be appreciable.

A. C. Hurley (Melbourne)

1554:

Franzinetti, C.; and Morpurgo, G. An introduction to the physics of the new particles. Nuovo Cimento (10) **6** (1957), supplemento, 469–802.

This review of the physics of hyperons and K-mesons was completed by the end of December, 1956. It gives one of the most complete and valuable surveys of the experimental situation as well as of the theoretical ideas current in the field up to that time. *A. Salam* (London)

1555:

Destouches, Jean-Louis. Le graviton et la gravitation en théorie fonctionnelle des corpuscules. C. R. Acad. Sci. Paris **245** (1957), 1518–1520.

A theory is outlined involving a system of non-linear equations for a particle of maximum spin 2 and describing the gravitational and electromagnetic fields.

N. Rosen (Haifa)

1556:

Feshbach, Herman; and Nichols, William. A wave equation for a particle of maximum spin one. Ann. Physics **4** (1958), 448–458.

A field equation, which describes two types of Bose particles of maximum spin 1, is investigated. It is shown that the two types of particles can have the same or different rest masses, but they must differ in either spin or parity. This work is an elaboration of an extremely interesting earlier paper by one of the authors [H. Feshbach, Phys. Rev. **98** (1955), 801–802].

S. N. Gupta (Detroit, Mich.)

1557:

Solov'ev, V. G. Equations for the Green's functions of a system of fundamental particles. Soviet Physics. JETP **6** (1958), 935–939.

"Green's functions for fundamental particles are introduced. On the basis of Lagrangians describing all strong interactions of mesons and baryons, closed systems of equations for the Green's functions are obtained in variational derivatives with respect to the external currents, in both three-dimensional and four-dimensional isotopic spin space." (Author's summary.)

A. Dalgarno (Belfast)

1558:

Kumar Pandit, Lalit. Electromagnetic properties of the nucleon and relativistic electron-proton scattering according to meson theory. Helv. Phys. Acta **31** (1958), 379–412.

It has been shown by the reviewer [S. N. Gupta, Phys. Rev. **77** (1950), 294–295] that in the treatment of virtual states containing two particles, it is possible to introduce a Lorentz-covariant variable of integration z, which is obtained from the energy-momentum four-vectors of the two particles. Using the reviewer's method of covariant integration and applying a cut-off to the invariant variable z, the author calculates the anomalous magnetic moments of nucleons and the elastic scattering of high-energy electrons by protons on the basis of the charge symmetrical pseudoscalar meson theory.

S. N. Gupta (Detroit, Mich.)

1559:

Zil'berman, G. E. Behavior of an electron in a periodic electric and a uniform magnetic field. Soviet Physics. JETP **5** (1957), 208–215.

This paper attacks the problem of the motion of electrons in metals or semiconductors under the influence of a magnetic field. There is an attempt to give a general proof of the quasi-classical approach to quantizing the "cyclotron orbits" which however suffers from the assumption shown incorrect by Kohn and Luttinger, that H can be constructed by setting $p \to p-(e/c)A$. Nonetheless some useful formalism is presented, as well as an interesting discussion of broadening of levels by the magnetic analog of the Zener effect.

P. W. Anderson (Raleigh, N.C.)

1560:

Eriksen, Erik. Foldy-Wouthuysen transformation in closed form for spin 1/2 particle in time-independent external fields. Norske Vid. Selsk. Forh., Trondheim **31** (1958), no. 7, 8 pp.

The Foldy-Wouthuysen transformation [Phys. Rev. **78** (1950), 29–36] splits the Dirac equation into two uncoupled equations of the Pauli type, for positive and negative energy states. This transformation in case of free particles is in closed form, but in the presence of interactions is an infinite series. Case [ibid. **95** (1954), 1323–1328; MR **16**, 656] subsequently found a closed form when the particle is in a time-independent magnetic field. The present paper goes one step further by finding the closed form in case of any time-independent external field. The transformation operator in question is $U=(1+Q)^{\frac{1}{2}}$, where $Q=\beta\Lambda-1$, and $\Lambda=H(H^2)^{-\frac{1}{2}}$, H being the Hamiltonian, and β the Dirac matrix for which $\beta_{11}=\beta_{22}=1$, $\beta_{33}=\beta_{44}=-1$. As an example the neutron in an electrostatic field is discussed.

M. J. Moravcsik (Livermore, Calif.)

1561:

Eriksen, Erik. Foldy-Wouthuysen transformation. Exact solution with generalization to the two-particle problem. Phys. Rev. (2) **111** (1958), 1011–1016.

This paper is almost identical to the one reviewed above. *M. J. Moravcsik*

1562:

Gartenhaus, S.; and Schwartz, C. Center-of-mass motion in many-particle systems. Phys. Rev. (2) **108** (1957), 482–490.

The centre of mass motion of a many-body system is taken care of by the introduction, not of a supplementary condition of constraint, but of a unitary operator of constraint. This operator applied to any state vector will see to it that the centre of mass can move freely. The action of this operator is illustrated by considering electric and magnetic multipole interactions and by solving exactly the equations for a system where all particles interact through harmonic oscillator forces. In the last case the present method sheds some light on the spurious states found by Elliott and Skyrme [Proc. Roy. Soc. London Ser. A **232** (1955), 561–566].

D. ter Haar (Oxford)

RELATIVITY

See also 1517.

1563:

Nagy, K. Relativistic equation of motion for spinning particles. Acta Phys. Acad. Sci. Hungar. **7** (1957), 325–339. (Russian summary)

Applicando un metodo dovuto a L. Infeld [cf. Bull. Acad. Polon. Sci. Cl. III, **3** (1955), 213–216; MR **17**, 201] per la derivazione delle equazioni del moto di particelle in un campo le cui equazioni sono lineari, l'A. ottiene le

equazioni del moto di particelle con spin con procedimento unitario più semplice di tutto quelli che sono stati fin qui proposti.

Successivamente l'A. studia il caso di una particella carica con momento magnetico. *G. Lampariello* (Roma)

1564:

Dragoş, L. Apropos d'un oscilateur de masse qui varie avec la vitesse. An. Univ. "C. I. Parhon" Bucureşti. Ser. Sti. Nat. **5** (1956), no. 12, 41–44. (Romanian. Russian and French summaries)

Un problème de la mécanique relativiste mène a l'équation $m_0\ddot{x}(1-\dot{x}^2/c^2)^{-3/2}+Kx=\varepsilon \sin \nu t$. Discussion d'une solution approximative; le cas spécial de résonnance. *O. Bottema* (Delft)

1565:

Hély, Jean. La notion de masse. Bul. Inst. Politech. Iaşi **4** (1949), 123–136.

A unification of the electromagnetic and gravitational fields is suggested in which the gravitational field is represented by a scalar potential. The conclusions of the paper rely on an assumption about the proper mass of the form

$$m_0^2 \cos(\pi/6)=m^2 \cos(\pi/6+\Phi),$$

where Φ is the gravitational potential.

C. W. Kilmister (London)

1566:

Gottlieb, Jean. Sur le champ relativiste d'une sphère matérielle de densité variable. C. R. Acad. Sci. Paris **245** (1957), 2016–2018.

The author presents a solution of Einstein's gravitational equations for a material sphere of non-uniform density. He intends to investigate the applicability of the solution to nuclei and elementary particles.

D. W. Sciama (London)

1567:

Bogolyubov, N. N.; and Parasyuk, O. S. On the subtractive formalism in multiplication of causal functions. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 585–610. (Russian)

1568:

Kurşunoğlu, Behram. Correspondence in the generalized theory of gravitation. Rev. Mod. Phys. **29** (1957), 412–416.

Dopo aver osservato l'incompletezza della teoria della relatività generale di Einstein e aver espresso la convinzione che un accordo con la teoria dei quanti della medesima teoria potrà essere raggiunto solo quando sarà stata formulata una conveniente generalizzazione, l'A. espone le sue idee riguardanti il modo di poter formulare la generalizzazione cercato. La ricerca non indica peró come effetivamente le particelle elementari possono essere rappresentate mediante soluzioni stazionarie delle equazioni del campo con forti fluttuazioni in regioni aventi le dimensioni di una costante fondamentale introdotta dall'A. in Phys. Rev. (2) **88** (1952), 1369–1379 [MR **14**, 805].

La difficoltà di costruire tali soluzioni sta nel fatto che le soluzioni esatte cercate debbono essere determinate dove le particelle sono situate e cioè in regioni in cui è essenziale la non-linearità delle equazioni.

Un' altra domanda alla quale la presente ricerca non risponde consiste nel vedere come si possa introdurre nella teoria la costante di Planck. *G. Lampariello* (Roma)

1569:

Straneo, Paolo. Essenza e utilizzazione attuale della relatività di Einstein. Confer. Sem. Mat. Univ. Bari no. 31 (1957), 24 pp. (1 photograph).

È questo il contenuto di una Conferenza dell'A. sui concetti basici della teoria della Relatività di Einstein e delle loro applicazioni alla fisica, all'astronomia e alla cosmologia.

Non sembra chiaro il pensiero dell'A. espresso al n. 4 della 1ª Parte riguardo al concetto del tempo proprio che, com' è noto, è stato introdotto da H. Minkowski e costituisce un invariante relativistico.

La pretesa assurdità denunziata dall'A. non esiste appunto a causa dell'invarianza relativistica. Nè si può pensare che il paradosso degli orologi sia superabile rimanendo nel dominio della teoria della Relatività speciale.

G. Lampariello (Roma)

1570:

Buchdahl, H. A. Variation of integrals and the field equations in the unitary field theory. Phys. Rev. (2) **104** (1956), 1142–1145.

L'A. mostra anzitutto che, postulando una certa relazione per il tensore asimmetrico fondamentale $g_{\mu\nu}$, è possibile evitare il metodo di Palatini [cfr. A. Einstein, The meaning of relativity, Princeton 1955, fifth edition, Appendix II; MR **17**, 907] per ottenere direttamente le derivate hamiltoniane di varie lagrangiane. Successivamente l'A. discute un insieme di equazioni del campo unitario che può essere considerato come il coniugato del sistema di equazioni di Einstein. Le equazioni di prima approssimazione dedotte da esse sono state studiate da Einstein e Straus [cfr. Ann. Math. **47** (1946), 731–741; MR **8**, 412] e da W. B. Bonnor [Proc. Roy. Soc. London Ser. A **209** (1951), 353–368; MR **13**, 695].

Una conseguenza di codeste equazioni approssimate che non è stata rilevata dagli Autori citati non sembra essere ammissibile. Si è allora condotti a proporre un altro sistema di equazioni che è soddisfacente dal punto di vista dell'approssimazione lineare, ma che sfortunatamente non è accettabile dal punto di vista dell'approssimazione quasi lineare. *G. Lampariello* (Roma)

1571:

Green, H. S. Spinor fields in general relativity. Proc. Roy. Soc. London. Ser. A. **245** (1958), 521–535.

The author revives Einstein's unified field theory based on distant parallelelism (Fernparallelismus) [Einstein, S.-B. Preuss. Akad. Wiss. **22** (1930), 401–402]. His starting-point is to define a connexion $\Delta^\nu{}_{\lambda\mu}$ in terms of a field of Dirac matrices γ_λ:

$$\Delta^\nu{}_{\lambda\mu}=\tfrac{1}{2}(\gamma^\nu\gamma_{\lambda,\mu}+\gamma_{\lambda,\mu}\gamma^\nu).$$

The covariant derivative of γ_λ with respect to $\Delta^\nu{}_{\lambda\mu}$ is assumed to vanish, which implies that $\Delta^\nu{}_{\lambda\mu}$ is a multiple of the unit matrix. It also follows that the curvature tensor formed from $\Delta^\nu{}_{\lambda\mu}$ vanishes, so that distant parallelism may be defined.

The electromagnetic potential is identified with the imaginary part of $\Delta^\nu{}_{\lambda\mu}$, and the field equations are derived from a quadratic Lagrangian, whose implications the author hopes to investigate.

The theory is then generalised by removing the restriction that the covariant derivative of γ_λ with respect to $\Delta^\nu{}_{\lambda\mu}$ is zero. This leads to the introduction of a pseudoscalar field, which is tentatively interpreted as the meson field responsible for nuclear forces.

{The reviewer is puzzled by the fact that in theories of this type the imaginary part of $\Delta^{\rho}_{\lambda\nu}$ is usually the gradient of a scalar, so that it does not give rise to an electromagnetic field. This led Weyl [Z. Physik **56** (1929), 330–352] and others to introduce a non-integrable similarity transformation of the γ_λ, which enables the electromagnetic field to be defined, but not in terms of the metrical properties of the space. It is not clear how the author is able to avoid this procedure.}

D. W. Sciama (London)

1572:

Pachner, Jaroslav. Die Bewegungsgleichungen der unitären Feldtheorie in der niedrigsten Annäherung. Ann. Physik (7) **1** (1958), 351–358.

The equations of motion in the author's unified field theory [Ann. Physik (6) **19** (1957), 353–368; **20** (1957), 368–380; (7) **1** [(1958), 110–115; MR **19**, 615, 1140] are worked out by the Einstein-Infeld-Hoffman method, and yield the Coulomb as well as the Newtonian force. It is stated that the author's Lagrangian differs from Bonnor's [Proc. Roy. Soc. London Ser. A. **226** (1954), 366–377; MR **16**, 755] only by a cosmological term, but that the identifications of electric and magnetic fields are exchanged. *F. A. E. Pirani* (Chapel Hill, N.C.)

1573:

Horváth, János; and Moór, Arthur. The theory of space developed for a general metric line-element space. Magyar Tud. Akad. Mat. Fiz. Tud. Oszt. Közl. **6** (1956), 53–72. (Hungarian)

A German version appears in Nederl. Akad. Wetensch. Proc. Ser. A. **58** (1955), 421–429, 581–587 [MR **17**, 544].

1574:

Zăgănescu, M. Quantification de l'espace à cinq dimensions. Acad. R. P. Romîne. Bul. Ştii. Secţ. Ştii. Mat. Fiz. **8** (1956), 715–722. (Romanian. Russian and French summaries)

"On montre qu'un schéma correct de l'espace-temps quantifié ne peut être obtenu que si l'on quantifie en plus une autre grandeur, qui peut être interprétée comme la cinquième coordonnée spatio-temporelle. Le réseau 5-dimensionnel est invariant par rapport à un certain groupe de rotations de l'espace à 5 dimensions. Dans le cas le plus simple, tridimensionnel, ce groupe est homomorphe du groupe modulaire, connu de la théorie des fonctions elliptiques.

Dans le cas du mouvement de l'electron dans un champ électromagnétique et un champ gravitationnel, on peut prendre le temps propre comme 5 coordonnée. La quantification du temps propre mène à une équation de mouvement analogue à celle qu'on obtient de la théorie du champ auxiliaire." (Resumé de l'auteur.)

A. G. Walker (Liverpool)

ASTRONOMY

See also 1513.

1575:

Jeffreys, Harold. Imperfections of elasticity in the small bodies of the solar system. Monthly Not. Roy. Astr. Soc. **117** (1957), 506–515.

L'A. esamina l'ipotesi dell' imperfetta elasticità di alcuni piccoli corpi del sistema solare, quali ad es. la Luna, Mercurio, ecc. per indagare quale sia l'influenza che tale imperfezione esercita sui periodi di rotazione dei corpi medesimi.

L'attenzione dell'A. è particolarmente rivolta alla Luna della quale è accertato lo smorzamento della variazione di latitudine verificantesi in un periodo di 14 mesi. I risultati della ricerca sono soddisfacenti per la Luna, mentre alcune incertezze sono inerenti al moto di Mercurio.

G. Lampariello (Roma)

1576:

Mihailović, Dobrivoje. Généralisation de certains résultats dans le problème du choc de trois corps. Univ. Beogradu. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. no. **12** (1957), 32 pp. (Serbo-Croatian. French summary)

The author himself points out in the introduction that in this paper he investigates the relation between two problems of Celestial Mechanics which up to now have been treated altogether separately. These are the problem of impact (collision) of three bodies and a problem studied by M. Milankovitch [Characteristics of motion in one specialized problem of three bodies, Glas Srpske Akad. Nauka. Od. Prirod-Mat. Nauka LXXIX, 1 (1909), 32] which this author names "the problem of bursting". This problem of Milankovitch refers to the motion of the center of gravity of three bodies originated by bursting of a single body.

The author shows that, under some additional assumptions concerning the problem of bursting, certain conclusions about the problem of impact of three bodies can be derived. *T. P. Anđelić* (Belgrade)

1577:

Kuzmin, G. G. On the gravitational potential of the galaxy and a third integral of the motion of the stars. Izv. Akad. Nauk Eston. SSR **2** (1953), 368–383.

In determining the component of the potential in the radial direction use is made of the curve of the circular velocity from which the values of the radial acceleration are calculated. The author does not have recourse to the rule for the rotation of the Galaxy obtained by Lindblad [see, e.g., Handb. d. Astrophysik **5/2**, 1933]

$$V_\theta = c_2 R/(c_1 + c_2 R^2,$$

since he considers that it has insufficient justification theoretically and does not completely represent the given observations.

Making use of the radial accelerations obtained in this way and assuming that the surfaces of equal density are spheroidal, the author calculates the values of the surface density of the Galaxy according to the formula

$$\delta = \frac{1}{2\pi} \int_R^\infty \frac{\mu(a)}{\sqrt{a^2 - R^2}} \frac{da}{a},$$

where a denotes the semi-axes of the spheroids included and μ is determined from the equation

$$-\frac{\partial \Phi}{\partial R} R = G \int_0^R \frac{\mu(a)da}{\sqrt{R^2 - a^2 e^2}}.$$

In the neighborhood of the sun δ is equal to 140 times the mass of the sun per cubic parsec and the gradient of the density logarithm is equal to −0.15 per kiloparsec.

On the basis of the data on the radial acceleration the mass of the Galaxy is estimated to be equal to 10^{11} times the mass of the sun and the velocity of escape in the region of the sun is equal to 365 km/sec.

A formula is obtained connecting the value of the potential for a certain z-coordinate with the space density

of the stars and the dispersion of velocities; here it is assumed that for separate subsystems of stars the distribution of z-components of velocity is approximately Gaussian

$$D=D_0 e^{\frac{\Phi-\Phi_0}{\sigma_z^2}}.$$

For regions near the galactic plane, this formula becomes

$$D=D_0 e^{-\frac{1}{2}(z/\zeta)^2},$$

where the paremeter ζ is connected with σ_z by the relation $\sigma_z/\zeta=C$,

$$C^2=-[\partial^2\Phi/\partial z^2]_{z=0}.$$

The value of C for proper motions of giant stars of spectral classes A and K is found to be equal to 56 ± 5 km/sec per kiloparsec. With this value for C in the Poisson equation connecting the magnitude of C and the Oort [J. H. Oort, Bull. Astr. Inst. Neth. 4 (1928), 269] constants A and B the density in neighborhoods of the sun is calculated to be equal to 0.05 ± 0.01 times the mass of the sun per cubic parsec.

In order to explain the triaxiality of the velocity-ellipsoid of the stars the author introduces [Publications of the Tartu Astronomical Observatory 32 (1953), 332–368] a third integral of the motion

$$I_3=(Rv_z-zv_R)^2+z^2v_0^2+z_0^2(v_z^0-2\Phi^*),$$

where v_R, v_z and v_0 are the components of velocity in the corresponding galactic system of coordinates, z_0 is a certain constant, and Φ^* is a function satisfying the equations

$$\frac{\partial\Phi^*}{\partial R}=\frac{z^2}{z_0^2}\frac{\partial\Phi}{\partial R}-\frac{Rz}{z_0^2}\frac{\partial\Phi}{\partial z},$$

$$\frac{\partial\Phi^*}{\partial z}=\left(1+\frac{R^2}{z_0^2}\right)\frac{\partial\Phi}{\partial z}-\frac{Rz}{z_0^2}\frac{\partial\Phi}{\partial R}.$$

The condition of consistency for these equations and consequently the condition of existence for the third integral, is written in the form

$$3\left(z\frac{\partial\Phi}{\partial R}-R\frac{\partial\Phi}{\partial z}\right)-(R^2+z_0^2-z^2)\frac{\partial^2\Phi}{\partial R\partial z}+Rz\left(\frac{\partial^2\Phi}{\partial R^2}-\frac{\partial^2\Phi}{\partial z^2}\right)=0.$$

The condition so derived also proves to be the condition for triaxiality of the ellipsoids for the velocity-distribution of the stars. Given the existence of this third integral the velocity ellipsoid proves to be inclined to the galactic plane, the angle of inclination being determined from the formula

$$\operatorname{tg} 2\alpha=\frac{2Rz}{R^2+z_0^2-z^2}.$$

Since z_0 is approximately constant over large regions of space, the assumption is made that to the first approximation z_0 is constant in the whole Galaxy. Then it is possible to consider the condition for the three axes of the velocity ellipsoids as a differential equation for the potential. The solution of this equation is given by the formula

$$\Phi=\frac{\phi(x_1)-\phi(x_2)}{x_1^2-x_2^2},$$

where ϕ is an arbitrary constant, while x_1^2 and x_2^2 are roots of the equation

$$\frac{R^2}{x^2-z_0^2}+\frac{z^2}{x^2}=1.$$

To show the usefulness of the expression obtained for the potential, the author examines a model of the Galaxy to which it leads. In this case the arbitrary function ϕ is given the following form

$$\phi=\Phi^0 z_0\sqrt{(x_0^2+x^2)},$$

where Φ^0 and x_0 are constants. The result is that $\sqrt{\Phi^0}=425$ km/sec, $z_0=3.6$ kiloparsecs, $x_0=0.6$ kiloparsecs. Since $x_0\ll z_0$, the potential on the galactic plane is given approximately by the expression

$$\Phi=\Phi^0\{1+(R^2/z_0^2)\}^{-\frac{1}{2}}.$$

Equipotential surfaces of the Galaxy are obtained in the form of spherical segments with centers on the galactic axis at points with the coordinates $+z_0$ and $-z_0$. Surfaces of equal density of a subsystem of short-period Cepheids are reasonably close to the corresponding equipotential surfaces. The cause of the considerable flattening of the surfaces of equal density of the Cepheids is considered by the author to lie in the triaxiality of the velocity distribution. The densities of the model constructed, when calculated using the Poisson equation, are everywhere positive. The deduction is made that the model of the Galaxy so obtained is admissible. Its fundamental usefulness consists, in the author's opinion, in the fact that, granting the existence of the third integral of the motion, this model explains the triaxiality of the velocity distribution within the framework of the theory of a stationary Galaxy.

There is a bibliography with 26 entries.

A. S. Šarov (RŽAstr 1955, no. 1487)

1578:

Sterne, Theodore E. The gravitational orbit of a satellite of an oblate planet. Astr. J. **63** (1958), 28–40.

An improved Hamiltonian is presented that approximates, more closely than can an ellipse, the gravitational motion of a close satellite of an oblate planet. Although the canonical constants have apparently no first-order secular perturbations, the mean anomalistic motion is changed because of them by a small and easily computed amount. The present unperturbed solution with the changed mean motion provides an ephemeris nearly as simply as the ellipse of the two-body problem but involves distinct and very accurate anomalistic, nodical, and sidereal periods. It may suffice for finding the ephemeris of an artificial earth satellite. With periodic first-order perturbations included, the results are expected to be considerably more accurate than a conventional ellipse with only first-order perturbations of its elements.

From the author's summary

1579:

Roberson, Robert E. Orbital behavior of earth satellites. I. J. Franklin Inst. **264** (1957), 181–202.

L'A. discute in questa 1ª Parte della sua ricerca le varie sorgenti di perturbazioni che influiscono sul moto orbitale di un satellite terrestre, rilevando l'incompletezza dei metodi della meccanica celeste classica e facendo un' accurata analizi dei contributi che le indagini moderne hanno portato di problemi astronautici specialmente dal punto di vista tecnico. Viene applicato in questo lavoro un metodo simile a quello delle perturbazioni di Hansen per la caratterizzazione dell'orbita approssimata del satellite.

G. Lampariello (Roma)

1580:

Roberson, Robert E. Orbital behavior of earth satellites. II. J. Franklin Inst. **264** (1957), 269–285.

In questo lavoro che è la continuazione di una precedente ricerca [≠1579 sopra] l'A. studia particolarmente la perturbazione esercitata su di un satellite intorno ad uno sferoide schiacciato dallo schiacciamento dello sferoide e dalla resistenza dell'atmosfera. Particolare attenzione è rivolta all'andamento dell'orbita in relazione alle condizioni iniziali e alla ricerca di un sistema di riferimento rotante rispetto al quale l'orbita sia periodica.

L'analisi dell'A. è limitata ai termini del prim'ordine nel parametro che caratterizza lo schiacciamento dello sferoide. *G. Lampariello* (Roma)

1581:

Kooy, J. M. J. On the application of the method of variation of elliptic orbit elements in case of a satellite vehicle. Astronaut. Acta **3** (1957), 179–214.

Il lavoro è diviso in due Parti. Nella 1ª Parte viene descritta la determinazione mediante osservazione degli elementi ellittici del moto Kepleriano tangente al moto di un satellite artificiale terrestre e viene applicato il metodo della variazione degli elementi ellittici del moto del satellite medesimo, quando si tenga conto dello schiacciamento terrestre, dell'attrazione del Sole e della Luna ed infine della resistenza aerodinamica nell'alta atmosfera.

Il confronto delle variazioni osservate e di quelle calcolate sulla base di diverse distribuzioni delle masse all'interno della Terra e di diverse densità atmosferiche deve poter fornire un controllo dell'attendibilità di codeste distribuzioni, nel caso che si voglia usare il satellite come strumento rivelatore della struttura del pianeta e dell'atmosfera.

La 2ª Parte del lavoro condotta seguendo criteri analoghi a quelli della 1ª Parte riguarda un satellite artificiale della Luna, senza equipaggio e può servire a rivelare la struttura interna lunare. *G. Lampariello* (Roma)

1582:

Kilmister, C. W. Eddington's statistical theory. II. The density-particle correlation. Rend. Circ. Mat. Palermo (2) **6** (1957), 33–50.

[For part I, see Bastin and Kilmister, same Rend. **5** (1956), 187–203; MR **18**, 782.] The author assumes that in a space-time $\bar{S}$ there is a vector-field $p^\alpha(x)$. To each point of $\bar{S}$ there is associated a point of a space-time S, the point with the same coordinates. The tensor field $T^{\alpha\beta}(x)$ in S is defined as

$$T^{\alpha\beta}(x)=\frac{c^4}{8\pi k\hbar^2}\frac{A}{N}\,p^\alpha p^\beta,$$

where c is the velocity of light, k is the gravitational constant, $\hbar$ is Planck's constant divided by 2π and N represents the number of particles represented by the vector field $p^\alpha(x)$.

The author discusses properties of the space-time S, assuming that it is a cosmological space. He further discusses the space-time $\bar{S}$, assuming {not proving as claimed} that it is conformal to S. The paper closes with a discussion of the so called quantum-theoretical treatment where the Einstein field equations are replaced by

$$\frac{A}{N}\,\psi_{;\alpha\beta}=G_{\alpha\beta}\psi,$$

where the semicolon denotes covariant differentiation in the space $\bar{S}$. This equation is considered as an equation for ψ in which $G_{\alpha\beta}$ is the Einstein tensor of the space S which is taken to be an Einstein universe. The significance of this function ψ is not made clear. It is not even stated in which space ψ is defined. *A. H. Taub* (Urbana, Ill.)

1583:

Kilmister, C. W.; and Tupper, B. O. J. Eddington's statistical theory. III. The uncertainty of the origin. Rend. Circ. Mat. Palermo (2) **6** (1957), 117–140.

In the previous paper [≠ 1582 above] the first author defined a scalar function ψ. In this paper this function is determined under more general assumptions. In particular the number of particles N is no longer required to be constant. {In spite of this, formulae derived on the assumption that N is constant are continually referred to.} The square of this function is assumed to give the probability of finding a particle in a given volume and related to the "uncertainty of the origin" of a particular rectangular cartesian coordinate system, which in turn is supposed to be related to the Hubble constant of a cosmological space-time. *A. H. Taub* (Urbana, Ill.)

1584:

Heckmann, O.; und Schücking, E. Bemerkungen zur Newtonschen Kosmologie. II. Z. Astrophys. **40** (1956), 81–92.

[For part I, see same Z. **38** (1955), 95–109; MR **17**, 545.]

The authors propose to replace the usual boundary condition "$\varphi=0$ at infinity" in Newtonian gravitational theory by the boundary conditions

$$\frac{\partial^2\varphi}{\partial x_i\partial x_j}-\tfrac{1}{3}\delta_{ij}\nabla^2\varphi=A_{ij}(t)-\tfrac{1}{3}\delta_{ij}\sum_{k=1}^{3}A_{kk}\text{ at infinity},$$

where the A_{ij} are arbitrary functions. They point out that these boundary conditions together with Poisson's equation determine the potential uniquely up to an additive term which is the potential of a uniform gravitational field.

{The proposed modification is easily seen to be untenable. In terms of the gravitational field $\mathbf{G}=\mathbf{grad}\ \varphi$ the proposed boundary condition reads "$\mathbf{G}=\mathbf{B}(t)+\mathbf{C}$ at infinity" where $\mathbf{B}(t)$ is an arbitrary vector field. But it is well known that when the divergence and curl of a vector field are specified in a region (in this case $\mathbf{div}\ \mathbf{G}=4\pi\rho$, $\mathbf{curl}\ \mathbf{G}=0$) the field itself is completely determined when either its tangential or its normal component is given on the surface bounding the region. The proposed boundary condition would specify both components on the surface of an infinite sphere, and hence could not in general be satisfied.} *D. Layzer* (Cambridge, Mass.)

1585:

Roberson, Robert E. Gravitational torque on a satellite vehicle. J. Franklin Inst. **265** (1958), 13–22.

In un lavoro dell'A. e di Tatistcheff [lo stesso J. **262** (1956), 209–214; MR **18**, 693] viene stabilita una funzione potenziale per un piccolo corpo rigido nel campo di uno sferoide schiacciato. In questo lavoro l'A. applica i risultati della precedente ricerca per il calcolo dei momenti delle forze gravitazionali rispetto agli assi principali del satellite. *G. Lampariello* (Roma)

1586:

Rosenstock, Herbert B. The effect of the earth's magnetic field on the spin of the satellite. Astronaut. Acta **3** (1957), 215–221.

Lo scopo del lavoro è di ricercare l'effetto sul moto di

un satellite nel campo magnetico terrestre. Più precisamente viene posto il problema di determinare la legge temporale della velocità angolare del satellite dovuta ad un campo magnetico uniforme ortogonale all'asse di rotazione.

Si trova che, supposto il satellite sferico e avente la forma di uno strato sferico, la velocità ω è del tipo

$$\omega(t)=\omega(0)\exp(-t/\tau),$$

essendo il tempo τ dipendente dalla natura del corpo e inversamente proporzionale al quadrato dell'induzione magnetica B.

Nel caso del campo geomagnetico si trova che τ è dell'ordine di dieci giorni. *G. Lampariello* (Roma)

GEOPHYSICS

See 1396, 1434, 1455, 1475, 1476, 1586.

OPERATIONS RESEARCH AND ECONOMETRICS

See also 1219, 1323, 1367, 1369, 1403.

1587:

Sargan, J. D. The distribution of wealth. Econometrica **25** (1957), 568–590.

The paper develops a model for the distribution of wealth. This distribution is supposed to be generated by a system of equations, continuous in time. Four separate causes of change in the distribution are considered, namely the setting up of new households, gifts, savings and capital gains, and the disappearance of existing households. In special cases the proposed model will result in an approximately log-normal distribution.

S. Malmquist (Stockholm)

1588:

***Vazsonyi, Andrew. Scientific programming in business and industry.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1958. xix+474 pp. $13.50.

This is an exposition, on an elementary level, of mathematical models and techniques which have recently been developed in theoretical studies of management science and resource allocation. The subjects covered include linear, convex, and dynamic programming; elementary game theory; statistical inventory control; and various more or less complex scheduling problems. The style is diffuse but ingratiating. It combines painstaking simplicity (the topics are all developed from simulated real-life numerical examples, and the reader is even told how to pronounce $f(x)$) with a certain amount of mathematical sophistication (mathematical symbolism is used freely, and full algebraic treatments, sometimes involving calculus, with assumptions stated carefully, are eventually given for most topics). The book is a valuable addition to the textbook literature of operations analysis. When used by non-mathematically trained students it may have to be supplemented by a considerable amount of drill and instruction in elementary mathematical symbolism. *J. H. Curtiss* (Providence, R.I.)

1589:

Frauenfelder, W. Bemerkungen und Anwendungen zur Theorie von Cantelli. Mitt. Verein. Schweiz. Versich.-Math. **58** (1958), 77–88.

Der Verf. gibt folgende Anwendungen des Fundamentalsatzes der Cantellischen Kapitalansammlungstheorie: Ermittlung der Prämiendifferenz bei Aenderung der Ausscheideintensitäten; Lidstonesche Formel und deren Erweiterung auf mehrere Leben; Näherungsformeln der Invalidenversicherung von M. Jacob und vom Referenten. *W. Saxer* (Zürich)

1590:

Guerrieri, Annibale. Sulla rendita vitalizia frazionata. Archimede **9** (1957), 186–191.

1591:

***Zwinggi, Ernst. Versicherungsmathematik. 2te Aufl.** Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, Mathematische Reihe, B. 1. Birkhäuser Verlag, Basel und Stuttgart, 1958. 258 pp. DM 37.30.

The first edition of this book (1945) was reviewed in MR **8**, 57. For the second edition, the work has been thoroughly revised and brought up to date. Moreover, a number of new sections have been added; amongst them are chapters on the elements of the continuous method, on the computation of additional premiums for increased mortality and on the variation of the basis of calculations in sickness insurance.

BIOLOGY AND SOCIOLOGY

See also 1597

1592:

***Bailey, Norman T. J. The mathematical theory of epidemics.** Hafner Publishing Co., New York, 1957. viii+194 pp. $6.75.

This important little book should be required reading for all graduate students of mathematical statistics and probability. The study of stochastic models of epidemics goes back to a pioneer paper of A. G. McKendrick [Proc. Edinburgh Math. Soc. **44** (1926), 98–130] and a parallel deterministic theory was initiated by McKendrick with W. O. Kermack at about the same time [Proc. Roy. Soc. London Ser. A **115** (1927), 700–721]. A large number of more complicated models (both deterministic and stochastic) have since been proposed by a number of writers and much mathematical ingenuity is demanded by and has been given to the study of the essentially non-linear problems involved. The author gives the first comprehensive account of the work that has been done in this field, and it is difficult to think how his task could have been better done. Estimation problems (for example, the measurement of latent and infectious periods [ch. 7]) are not forgotten, but to the mathematical reader the most interesting parts of the book are those in which a model is defined and an attempt is then made to give a qualitative description of its behaviour. Threshold phenomena abound (four distinct 'threshold theorems' appear in the index); even in the deterministic case these lead to significant mathematical problems and many of them are still 'open'. There are research problems here for

everyone. In particular the non-linear differential equation (4.46) on page 33 deserves and has not yet been given serious attention. *D. G. Kendall* (Oxford)

INFORMATION AND COMMUNICATION THEORY

See also 1519, 1599

1593:

Goldschmied, Bedřich. On the theory of telegraph transmission. Apl. Mat. **3** (1958), 170–189. (Czech. Russian and English summaries)

By use of the Schrödinger equation, the author shows that the best pulse shape for telegraphy is the Gaussian, where 'best' is defined, in terms of the "uncertainty principle" of Fourier analysis, as minimizing $\Delta f \cdot \Delta t$. This result is well-known and has a simpler proof. An analogous result for the best band-limited pulse is stated.

V. E. Beneš (Murray Hill, N.J.)

1594:

***Feinstein, Amiel. Foundations of information theory.** McGraw-Hill Electrical and Electronic Engineering Series. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1958. x+137 pp. $6.50.

"The intention of the author in writing this book is to provide a concise but rigorous exposition of the fundamentals of the mathematical theory of information. Since the basic work of C. E. Shannon appeared in 1948, a good deal of work has been done in this field. Nevertheless there exists at present no single reference to which one interested in this subject can turn for a presentation which is up to date and yet reasonably complete. It is hoped that the present work may help to fill this gap." (From the author's preface.)

In chapter 1 the author gives D. K. Faddeev's [Uspehi Mat. Nauk (N.S.) **11** (1956), no. 1 (67), 227–231; MR **17**, 1098] derivation of the Shannon formula $H(p_1, \cdots, p_n) = \sum_{i=1}^{n} p_i \ln(1/p_i)$ for the entropy of finite probability fields $(p_1, \cdots, p_n)$ from the axioms (1) $H(p, 1-p)$ is continuous on $0 \leq p \leq 1$, (2) $H(p_1, \cdots, p_n)$ is symmetric in $p_1, \cdots, p_n$, and (3) if $p_n = q_1 + q_2 > 0$ then $H(p_1, \cdots, p_{n-1}, q_1, q_2) = H(p_1, \cdots, p_n) + p_n H(q_1/p_n, q_2/p_n)$. In chapter 2, joint and conditional entropies are defined, and various inequalities based on the convexity of $x \ln x$ and $\ln(1/x)$ are derived (e.g., the entropy of a finite probability field is not increased by conditioning.) In chapter 2 appears also the basic "noiseless coding theorem", to the effect that the expected average number of D-ary digits per trial needed to specify a sample from a distribution $(p_1, \cdots, p_n)$ is $H(p_1, \cdots, p_n)/\ln D$. Chapters 3 through 6 contain the central presentation of the book, the McMillan [Ann. Math. Statist. **24** (1953), 196–219; MR **14**, 1101], Feinstein [Trans. I.R.E. **PGIT-4** (1954), 2–22; MR **19**, 516], Hinčin [Uspehi Mat. Nauk (N.S.) **11** (1956), no. 1 (67), 17–75; MR **17**, 1098] proof of the Shannon coding theorem for the noisy channel (in the discrete parameter, stationary, finite memory case.) This theorem asserts that by using sufficiently elaborate error correcting codes it is possible to signal essentially without error over an error making (noisy) communication channel at a positive rate, the "capacity" of the channel. In chapter 7 the author gives an account of recent work on error rates and error correcting codes for the binary symmetric channel. A "Remarks" section at the end of each chapter gives mathematical complements and details. Chapter 5 opens with an eleven page précis of the measure-theoretic notions used in chapters 5 and 6 (Borel fields, measures, integration, the Radon-Nikodym theorem, product measures). *S. P. Lloyd* (Murray Hill, N.J.)

1595a:

Perez, Albert. Mathematical theory of information. I. Apl. Mat. **3** (1958), 1–21. (Czech)

1595b:

Perez, Albert. Thèorie mathématique de l'information. II. Apl. Mat. **3** (1958), 81–105. (Czech. Russian and French summaries)

This article, appearing in two parts, is simultaneously a review of the basic concepts of information theory, and a presentation of generalizations of the chief theorems to discrete-parameter processes with arbitrary alphabets. Proofs are only sketched, and the reader is referred to the author's previous work. The first part contains a well-conceived analogy of information transfer to observation in statistical mechanics, and a discussion of a notion of stochastic distinguishability, partly due to Kolmogorov and fundamental to Feinstein's theorem. A notion of generalized entropy of one measure with respect to another is introduced, on which generalized forms of the usual notions of entropy, conditional entropy, etc., are based.

In the second part general forms of the theorems of McMillan, Feinstein, and Shannon are stated, for arbitrary alphabets and under assumptions of absolute continuity. The development largely follows Hinčin's monograph, [Uspehi Mat. Nauk (N.S.) **11** (1956), no. 1 (67), 17–75; MR **17**, 1098]. A closing section on continuous signals and applications refers to the sampling theorem as Kotělnikov's theorem, although the result in question is due to Cauchy [C. R. Acad. Sci. Paris **12** (1841), 283–298], and was rediscovered by Whittaker [Proc. Roy. Soc. Edinburgh, **35** (1915), 181–194].

V. E. Beneš (Murray Hill, N.J.)

1596:

Varšamov, R. R. The evaluation of signals in codes with correction of errors. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 739–741. (Russian)

Let D be the space of the $N = 2^n$ sequences, each of which consists of n elements 0 or 1. Let S be a set of s elements in D, such that any two members of S differ in at least $d = 2r+1$ places, where r is a positive integer. Problem: How large is $M = \sup_{S \subset D} s$? Hamming [Bell System Tech. J. **29** (1950), 147–160; MR **12**, 35] proved that $M \geq N/(1 + C_n^1 + \cdots + C_n^{d-1})$, where $C_n^p = \dfrac{n!}{p!(n-p)!}$. The author proves that $M > N/(1 + C_{n-1}^1 + \cdots + C_{n-1}^{d-2})$. This problem has application to the theory of error correcting codes. *J. Wolfowitz* (Ithaca, N.Y.)

1597:

Golomb, S. W.; Gordon, Basil; and Welch, L. R. Comma-free codes. Canad. J. Math. **10** (1958), 202–209.

A set D of k-letter words, from an n-letter alphabet, is called a comma-free dictionary if, when two such words are adjoined, no set of k consecutive letters, other than those in the chosen words, form a word in D. Let $W_k(n)$ be the number of words in a comma-free dictionary. The following results are proved: (1) $W_k(n) \leq (1/k) \sum \mu(d) n^{k/d}$, where the summation is over all divisors d of k, and $\mu(d)$ is the Möbius function. (2) $W_k(n)$ attains its u.b. for all

odd $n \leqq 15$. (It is conjectured that the result holds for all odd n.) (3) $W_2(n)=[n^2/3]$. (4) For even k, the u.b. is not attained by $W_k(n)$, for $n>3^{k/2}$. (5) $\lim_{n\to\infty}\{W_k(n)/n^k\}=\alpha_k$ exists. (6) For odd k, $\alpha_k=1/k$. (7) For even k, $1/ek<\alpha_k\leqq 1/k$.

A previous paper on this subject by Crick, Griffith and Orgel [Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 416–421; MR **19**, 234] indicated an application to the theory of transfer of genetic information. *P. Armitage* (London)

CONTROL SYSTEMS

1598:

Dostupov, B. G. Approximate determination of probability characteristics of the outputs of automatic nonlinear control systems. Avtomat. i Telemeh. **18** (1957), 999–1009. (Russian. English summary)

The author describes and illustrates an approximate practical method for obtaining moments $M[X^p(t)]$ of the solutions of a system

$$(*)\quad \frac{dX_\nu}{dt}=f_\nu(X_1,\cdots,X_n,V_1,\cdots,V_m,t)\ (\nu=1,2,\cdots,n),$$

where the V_i are random parameters with given moments $\mu_{r_1\cdots r_s}=M[V_{r_1}V_{r_2}\cdots V_{r_s}]$. The method depends on the feasibility of obtaining explicit solutions $x_s(t)$ of (*) when the V_i are replaced by the specific set of real numbers $V_i=\xi_{is}$ ($i=1,2,\cdots,m$; $s=1,2,\cdots,N$). N is chosen sufficiently large so that an approximate set of probabilities α_s can be assigned to the points $(\xi_{1s},\cdots,\xi_{ms})$ in such a way as to produce good agreement with the quantities $\mu_{r_1r_2\cdots r_s}$. Then $M[X^p]\approx\sum_{s=1}^{N}\alpha_s x_s^p$. {There appears to be a misprint in eq. (6.6).}

E. Reich (Minneapolis, Minn.)

1599:

Tihonov, V. I. Effect of fluctuations in simplest parametric systems. Avtomat. i Telemeh. **19** (1958), 717–724. (Russian. English summary)

Consider a system characterized by the differential equation

$$\frac{dy}{dt}+\xi(t)y=\eta(t),$$

in which $\{\xi(t)\}$ and $\{\eta(t)\}$ are stationary Gaussian processes. The author obtains explicit expressions for $E\{y(t)\}$ and $E\{y(t_1)y(t_2)\}$ in terms of the initial value of $y(t)$ at $t=t_0$, $E\{\xi(t)\}$, $E\{\eta(t)\}$, and the covariance functions $R_{\xi\xi}$, $R_{\xi\eta}$ and $R_{\eta\eta}$. As an illustration, the case $\xi(t)=$constant is considered, and the asymptotic stability (in mean square) of such systems is investigated. In particular, it is found that under certain stated conditions one can have $E\{y(t)\}\to 0$ and $E\{y^2(t)\}\to\infty$ as $t\to\infty$.

L. A. Zadeh (New York, N.Y.)

1600:

***Ku, Y. H. Analysis and control of nonlinear systems. Nonlinear vibrations and oscillations in physical systems.** The Ronald Press Company, New York, 1958. vii+360 pp. $10.00.

This work is addressed primarily to electrical engineers and designers of control systems. The author treats in detail the standard methods of solution of nonlinear differential equations arising in the analysis of electrical as well as mechanical systems. In the case of higher order equations, the author employs his method of space trajectories [J. Franklin Inst. **259** (1955), 115–131; MR **16**, 701] which involves essentially a step-by-step graphical determination of trajectories in the $(x^{(i)}, x^{(j)})$ planes, where $x(t)$ is a solution of nth order differential equation, $x^{(i)}(t)=d^ix/dt^i$, and $i, j=1,\cdots,n$. The method is illustrated by many examples, but its exposition is heuristic and there is no analysis of the effect of cumulative errors due to the replacement of differentials by divided differences.

The last two chapters (about one-third of the book) are given over to applications of the theory to the analysis of multi-loop nonlinear networks and nonlinear servomechanisms. Here the stability theory of control systems is treated very incompletely and, in particular, there is no mention of the recent work of the Russian school. The book closes with a very extensive bibliography covering the period 1860–1957 and containing 735 entries.

L. A. Zadeh (New York, N.Y.)

1601:

Bedel'baev, A. K. On the stability of non-stationary motion of a class of self-regulating systems. Akad. Nauk Kazah. SSR. Trudy Sektor. Mat. Meh. **1** (1958), 151–159. (Russian)

1602:

Rohleder, Hans. Über eine Theorie einiger Klassen von elektrischen Schaltungen. Z. Math. Logik Grundlagen Math. **3** (1957), 225–291.

This study is motivated by the problem of finding an algebra which will do for relay circuits — i.e. circuits containing windings of certain relays as well as contacts — what Boolean algebra does for series-parallel contact circuits. The author does not solve this general problem, but he obtains some partial solutions which apply to the following types of cases: a) contact networks, i.e. circuits containing contacts of externally actuated relays, and b) certain special kinds of electronic and relay circuits (in the above sense). For his algebra the author uses a propositional logic with three values 0, $\frac{1}{2}$, 1; operations $\wedge$, $\vee$ $\sim$, $\vdash$, $\leftrightarrow$ where $a\wedge b$ is the minimum of a and b, $a\vee b$ the maximum of a and b, $\sim a$ is $1-a$, $a\leftrightarrow b$ is 1 when $a=b$ and is otherwise 0, and $\vdash a$ is $a\leftrightarrow 1$. The third value $\frac{1}{2}$ is used when the state of a circuit element is not determined by the imposed conditions. Each of the types of circuits considered leads to a set of equivalences of the form

$$(1)\qquad x_i\leftrightarrow H_i(a_1,\cdots,a_m,x_1,\cdots,x_n),$$

where the a_i stand for circuit elements externally controlled and take the values 0, 1, only; whereas the x_i are elements whose state is to be determined by the circuit, except that in certain cases, where H_i is of the form $x_i\vee H_i$, some of such x_i may be required to be 1. The H_i, for these circuits, are Boolean expressions, i.e., they are constructed without use of $\vdash$, $\leftrightarrow$, and thus either are identically 1 or 0, or take the value $\frac{1}{2}$ when all variables are $\frac{1}{2}$. Under these circumstances the equivalences (1) always have a solution. A solution such that a maximal set of the x_i have the value $\frac{1}{2}$ is called a Hauptlösung; this solution can be obtained by a process of finite iteration. Under given electrical conditions an element x_i is actuated just when $x_i=1$ in the Hauptlösung; and two systems of the same type are electrically equivalent just when they have the same Hauptlösung under all conditions. Various transformations leaving the Hauptlösung invariant are considered. The reviewer is quite unable to appraise all this from the standpoint of circuit design; but it is certainly an interesting application of three valued propositional algebra. The article is self-contained; in

particular it contains all the theory of three-valued algebra which is necessary for its understanding.

H. B. Curry (University Park, Pa.)

1603:

Rohleder, Hans. Über die Verwendung eines Aussagenkalküls bei der Synthese von Rechenwerken. Z. Math. Logik Grundlagen Math. 3 (1957), 292–302.

This is a sequel to the paper reviewed above. In the case of a circuit used for a calculation process (Rechenverfahren) the solution is unique and coincides with the Hauptlösung. It is shown by examples that a calculation (Rechenoperation) can be carried out by different calculation processes which do not have the same Hauptlösung. This is because the auxiliary quantities in the two processes may be different. Examples of transformation rules (Umformungsregeln) are given for transforming one such process into another which is equivalent so far as that calculation is concerned. It is alleged that these rules are complete; but the indications given in support of this are brief and indirect. At the end of the paper it is indicated that further studies are in progress.

H. B. Curry (University Park, Pa.)

HISTORY AND BIOGRAPHY

See also 830, 1325.

1604:

Freudenthal, Hans. Foucaults Pendelversuch in der Differentialgeometrie. Math.-Phys. Semesterber. **5** (1957), 230–238.

German version of Simon Stevin **31** (1957), 49–60 [MR **19** 477]. The paper originated as an address to science students in the Netherlands in commemoration of Foucault (1951). *D. J. Struik* (Cambridge, Mass.)

1605:

Cassina, Ugo. Sur l'histoire des concepts fondamentaux de la géométrie projective. Les Conférences du Palais de la Découverte, Série D, No. 50. Université de Paris, 1957. 36 pp. 140 francs.

This historic survey of the development of the concepts of projective geometry and its axiomatic foundations leads first from the "Brouillon project" by Desargues to Poncelet's "principle of the law of continuity" and the notions of point, line and plane at infinity. Then follows the evolution of projective geometry through the work of Cayley, Bolyai, Lobatschefski to Von Staudt and Klein. After an evaluation of Peano's postulates the author, influenced by ideas of Pasch, gives an outline of his own contributions to axiomatics [cf., e.g., the author's paper in Atti I Congresso di Studi Metodologici, Torino, 1954, pp. 155–161]. *S. R. Struik* (Cambridge, Mass.)

1606:

Conte, Luigi. G. Bernoulli, G. C. de' Toschi di Fagnano e la sfida di Brook Taylor. Bul. Inst. Politech. Iaşi **4** (1949), 36–53.

A discussion, in the historical setting of hostility between the Newtonians and the Leibnitzians, of the answers given by Bernoulli and de' Toschi di Fagnano to Taylor's challenge, namely to integrate the differential

$$z^{\delta q\lambda-1}dz/(e+fz^q+gz^{2q})$$

where δ is an integer and λ is any positive power of 2. The point to the challenge lies in the case where the quadratic in the denominator has complex roots.

1607:

Fichera, Gaetano. Cenni sui problemi di analisi matematica contemporanea. I. Produzione italiana nel campo dell'analisi matematica durante il periodo 1940-1945. Bul. Inst. Politech. Iaşi **4** (1949), 63–107.

There are 8 headings: Didactic works, analytic functions of a complex variable, of several complex variables, theory of sets and of functions, ordinary differential equations, partial differential equations, calculus of variations, and integral equations and functional analysis. There is an extensive bibliography of over 100 entries.

1608:

Biermann, Kurt R. From the history of the calculus of probability. Pokroky Mat. Fys. Astr. **2** (1957), 31–35. (Czech)

A translation from the German of Wiss. Ann. **5** (1956), 542–548 [MR **18**, 453].

1609:

Bespamyatnih, N. D. Studies in number theory in Russia in the 19th century. Grodnen. Gos. Ped. Inst. Uč. Zap. Ser. Mat. **2** (1957), 3–42. (Russian)

There are six chapters, with the headings: Survey of the number-theoretic works of P. L. Čebyšev, Distribution of primes in the natural numbers, Distribution of primes in arithmetic progressions, Sieve methods, Development by foreign mathematicians of the number-theoretic ideas of Čebyšev, Divisibility. There is a bibliography of 70 entries.

1610:

Pavlović, D. Contribution à la biographie de Marin Getaldić. Srpska Akad. Nauka. Zb. Rad. **55**, Mat. Inst. **6** (1957), 77–87. (Serbo-Croatian. French summary)

On the basis of documents recently discovered in Ragusa, Yugoslavia, the exact date of Ghetaldi's birth is now determined and more precise information is obtained about his education, services to the state, visits to Rome and Constantinople, and his family and possessions.

1611:

Anonymous. Obituary: Vasilii Vasil'evič Popov. Akad. Nauk BSSR Trudy Inst. Fiz. Mat. **1956**, no. 1, 3–10. (Russian)

A general and scientific biography with a bibliography of 35 entries.

1612:

Kořínek, Vladimír; and Vyčichlo, František. Academician Vojtěch Jarník on his sixtieth birthday. Pokroky Mat. Fys. Astr. **3** (1958), 1–8 (1 plate). (Czech)

A general and scientific biography with a photograph.

1613:

Aleksandrov, P. S.; and Gluškov, V. M. Aleksandr Gennadievič Kuroš (on the fiftieth anniversary of his birth). Uspehi Mat. Nauk (N.S.) **13** (1958), no. 1(79), 217–224 (1 plate). (Russian)

A biography with a photograph and a bibliography of 43 entries.

1614:
Norden, A. P.; Rozenfel'd, B. A.; and Yaglom, I. M. Petr Konstantinovič Raševskiĭ (on the fiftieth anniversary of his birth). Uspehi Mat. Nauk (N.S.) 13 (1958), no. 1(79), 225–231 (1 plate). (Russian)
A biography with a photograph and a bibliography of 58 entries.

1615:
Hasse, Helmut. Wissenschaftlicher Nachruf auf Hermann Ludwig Schmid. Math. Nachr. 18 (1958), 1–18.
An obituary, with special emphasis on Schmid's scientific work. There is a bibliography of 24 entries.

1616:
***Крылов, А. Н. Собрание трудов академика А. Н. Крылова. X. Вибрация судов. [Collected works of academician A. N. Krylov. X. Vibration of ships.]** Edited by Yu. A. Šimanskiĭ. Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1948. 402 pp. (1 insert) 25 rubles.
This book was first published in 1936 as a textbook in shipbuilding for Technical Schools.

GENERAL

1617:
***Magner, Thomas F. Manual of scientific Russian.** Burgess Publishing Co., Minneapolis, Minn., 1958. iii+101 pp. (1 insert) $4.60.
The writer has dispensed with the slow, shock-avoiding approach of the multi-purpose textbook. The structure of the Russian language has been described as briefly and efficiently as possible, so that the learner may begin at an early stage to read the Russian texts contained in the manual. (From the introduction.)
Passages for study are given from many different sciences, including mathematics (differential equations).

1618:
***Russian-English glossary of acoustics and ultrasonics.** Consultants Bureau, Inc., New York, 1958. iii+171+xxiii pp. $10.00.
Over 10,000 terms from the fields of acoustics, electroacoustics and ultrasonics.

1619:
***Russian-English glossary of solid state physics.** Consultants Bureau, Inc., New York, 1958. iii+90 pp. $10.00.
This glossary contains approximately 4,000 terms compiled from the most recent issues of the Journal of Experimental and Theoretical Physics, the Journal of Technical Physics and the Physics section of the Proceedings of the USSR Academy of Sciences.

1620:
***James, Glenn (Editor). The tree of mathematics.** The Digest Press, Pacoima, Calif., 1957. xvii+403 pp. $6.00.
Covering nearly 30 branches of mathematics, the different authors, in a space of 10 to 20 pages each, acquaint the student of high school or elementary college age with fundamental concepts and procedures. Included are calculus and analytic geometry, vectors, differential equations, number theory, differential, non-euclidean and projective geometry, topology, functions of a real and complex variable, series, sets, probability, dynamic programming and games. Among the authors we find many well-known mathematicians of the U.S.A.; also H. S. M. Coxeter of Toronto, and M. Fréchet of Paris. Support in editing came from E. F. Beckenbach and D. H. Hyers. Problems, and occasionally bibliography, are added to the text. Many chapters have a short introduction in non-technical language. An index interrelates the information of the divergent fields.

S. R. Struik (Cambridge, Mass.)

AUTHOR INDEX

(*Continued from cover 2*)

Curzio, M. ... 905
Dahmen, G. ... 1108a-b
Dalecki˘ı, Yu. L. ... 1199
Danzer, L. ... 1283
Davydov, N. A. ... 1138
Day, M. M. ... 1187
Deleanu, A. ... 1296
De Munter, P. ... 1391
Dennis, S. C. R. ... 1460
Deny, J. ... 1119
Desgranges, J. ... 814
Des Raj ... 1377
Destouches, J.-L. ... 1555
Dieudonné, J. ... 930, 931, 932
Dikiĭ, L. A. ... 1414
Dix, C. H. ... 1434
Dixmier, J. ... 893, 924, 1234
Dixon, W. J. ... 1376
Djerasimović, B. ... 850
Dlab, V. ... 901
Dlin, A. M. ... 1398
Dobrescu, E. V. ... 939
Dobrušin, R. L. ... 1352, 1362
Domb, C. ... 1128
Donskaya, L. I. ... 1013
Dostupov, B. G. ... 1598
Dou S. I., A. ... 1326
Downing, H. H. ... 944
Drăgan, I. ... 1081
Dragoș, L. ... 1564
Dubnow, J. S. ... 1251
Dubois, D. W. ... 844
Dubovickiĭ, A. Ya. ... 942
Dvoretzky, A. ... 1364
Dyson, F. J. ... 1537
Egerváry, E. ... 834
Eggleston, H. G. ... 1280, 1281
Ehrman, J. B. ... 1239
Eĭdel'man, S. D. ... 1097, 1098
Elazar, S. ... 1070
Elianu, I. P. ... 1120
Emersleben, O. ... 1429
Emmons, H. W. ... 1485
Engel, W. ... 879
Eremin, S. A. ... 1143
Erdös, P. ... 1364
Eriksen, E. ... 1560, 1561
Ešukov, L. N. ... 1027
Evgrafov, M. A. ... 987
Ewing, W. M. ... 1475
Eyring, H. ... 1509
Ězrohi, I. A. ... 1217
Fabian, V. ... 1424
Facciotti, G. ... 863
Faddeev, L. D. ... 1025
Faedo, S. ... 1247
Fage, M. K. ... 1011a-b
Fagen, R. E. ... 1365
Falgas, M. ... 985
Fano, G. ... 1261
Faure, R. ... 1039, 1445
Feenberg, E. ... 1543
Feinstein, A. ... 1594
Fernandes Costa, M. A. ... 1388
Ferrarese, G. ... 1462
Feshbach, H. ... 1556
Feys, R. ... 817
Fichera, G. ... 1607
Fiedler, M. ... 1252
Findlay, G. D. ... 888
Fisher, M. E. ... 1128
Fišman, K. M. ... 837
Fleksser, M. S. ... 1069
Fleming, W. H. ... 1277
Flügge, W. ... 1464
Folner, E. ... 1237
Ford, L. R., Jr. ... 1403
Fort, T. ... 1125
Fortet, R. ... 1358
Franckx, E. ... 1405
Franklin, P. ... 956
Franzinetti, C. ... 1554
Frauenfelder, W. ... 1589
Freeman, N. C. ... 1487
Freud, G. ... 1147
Freudenthal, A. M. ... 1453
Freudenthal, H. ... 1604
Froda, A. ... 938
Froese, C. ... 1545
Frum-Ketkov, R. L. ... 1305
Fuchs, A. ... 1357
Fulkerson, D. R. ... 1403
Furuya, S. ... 1406
Fyadzenka, A. S. ... 1329
Gagaev, B. M. ... 1186
Gaĭnov, A. T. ... 890
Gamkrelidze, R. V. ... 1016
Ganea, T. ... 1315
Gantmacher, F. R. = Gantmaher, F. R.
Gantmaher, F. R. ... 1037
Garnir, H. G. ... 1066a-b
Gartenhaus, S. ... 1562
Gartštein, B. N. ... 1346
Gastinel, N. ... 1404
Gavurin, M. K. ... 1242
Gehring, F. W. ... 1112
Gel'fand, I. M. ... 1061
Gel'fer, S. A. ... 981
Gelfond, A. O. ... 1121
Gel'man, A. E. ... 1017
Genuys, F. ... 1436
Geronimus, Ya. L. ... 1173
Getoor, R. K. ... 1359
Gheorghiev, G. ... 1488
Ghermănescu, M. ... 1127, 1129
Ghika, A. ... 1222
Ginzburg, I. P. = Ginzburg, Yu. P.
Ginzburg, Yu. P. ... 1040, 1203
Glinskaya, N. N. ... 1419
Gloden, A. ... 835
Gluškov, V. M. ... 1136
Gobert, J. ... 1066a-b
Godeaux, L. ... 1327
Goldberg, M. ... 1438
Goldberg, V. N. ... 1226
Gol'dfain, I. A. ... 1178
Goldhagen, E. ... 1104a-b
Goldschmied, B. ... 1593
Golomb, S. W. ... 1597
Goodman, L. A. ... 1356, 1380
Gordon, B. ... 1597
Görtler, H. ... 1498
Goto, K. ... 1517
Gottlieb, J. ... 1566
Gourdin, M. ... 1544
Green, H. S. ... 1571
Green, M. S. ... 1446
Grenander, U. ... 1349
Griesel, H. ... 1130
Griffiths, H. B. ... 1294
Gröbner, W. ... 877
Grothendieck, A. ... 1194, 1195
Grünbaum, M. ... 1253
Guerrieri, A. ... 1590
Guest, P. G. ... 1392
Gupta, Hansraj ... 843
Gupta, Subhas Chandra ... 1492
Gusak, A. A. ... 1282
Halmoš, A. ... 1220
Halperg, C. J. A., Jr. ... 1225
Halton, J. H. ... 1400
Handscomb, D. C. ... 1400
Hardy, G. H. ... 828
Hare, A. ... 1511
Harish-Chandra ... 925
Harper, P. G. ... 1447
Hartley, H. O. ... 1380
Harybin, A. E. ... 1397
Hasse, H. ... 1615
Hattori, A. ... 854a-b
Havel, V. ... 1259
Hazanov, M. B. ... 852, 1272
Heckmann, O. ... 1584
Heins, M. ... 975
Heller, I. ... 899
Helms, L. L. ... 1350
Hély, J. ... 1565
Hemmer, P. C. ... 1443
Henrici, P. ... 1410
Hermes, H. ... 1343
Herstein, I. N. ... 913
Hill, J. A. F. ... 1502
Hilton, P. J. ... 940
Hölder, E. ... 969
Hornich, H. ... 980, 1052
Horváth, J. ... 1024, 1573
Hua, L. K. ... 829
Huber, A. ... 970
Hunter, R. P. ... 1285
Huppert, B. ... 904
Hvedelidze, B. V. ... 963
Ignat'eva, R. P. ... 839, 840
Ikoma, K. ... 958
Inönü, E. ... 927
Inoue, M. ... 1116
Integrals of Airy functions ... 1428
Irwin, G. R. ... 1454
Isbell, J. R. ... 923
Ishi, T. ... 947, 948
Itô, H. ... 1520
Ivanov, V. K. ... 989
Ivašev-Musatov, O. S. ... 1164
Iwabori, C. ... 1201
Jaffard, P. ... 858, 859
Jakimovski, A. ... 1140
James, G. ... 1620
Jans, J. P. ... 886
Jardetzky, W. S. ... 1475
Jasper, S. J. ... 944
Jeffreys, H. ... 1575
Jennings, S. A. ... 906
Jevons, W. S. ... 815
Jiřina, M. ... 1361
Joshi, G. H. ... 1521
Jůza, M. ... 1291
Kačmaž, S. ... 1148
Kaczmarz, S. = Kačmaž, S.
Kadašević, Yu. I. ... 1480
Kähler, E. ... 1107
Kakutani, S. ... 1364
Kane, T. R. ... 1463
Kantorovič, L. V. ... 1219
Kargapolov, M. I. ... 914
Karlin, S. ... 1363
Kästner, S. ... 1477
Kas'yanyuk, S. A. ... 982
Kautský, J. ... 1099
Kawada, Y. ... 894
Keller, J. B. ... 1507
Kemperman, J. H. B. ... 1123
Kervaire, M. A. ... 1337
Khalatnikov, I. M. ... 1546
Khamis, S. H. ... 1377
Kikodze, E. B. ... 919
Kilmister, C. W. ... 1582, 1583
Kilpi, Y. ... 1172
Kiyohara, M. ... 1162
Klee, V. L., Jr. ... 1200
Kockel, B. ... 1523
Kolchin, E. ... 1109
Kollbrunner, C. F. ... 1465
Kondraševa, I. L. ... 1422
Konoplev, V. P. ... 1034
Konstantinescu, F. ... 943
Kooy, J. M. J. ... 1581
Kořínek, V. ... 1612
Korvasová, K. ... 1411
Koščelev, A. I. ... 1481
Kostomarov, D. P. ... 1047
Kostrikin, A. I. ... 898
Kostyučenko, A. G. ... 1068
Kotlyar, Ya. M. ... 1494
Kotzig, A. ... 1321
Koval', P. I. ... 1126
Kracovskaya, E. S. ... 1133
Krasnosel'skiĭ, M. A. ... 1243
Krein, M. G. ... 1018
Kreyszig, E. ... 1062
Krohová-Santavá, S. ... 1019
Krylov, A. N. ... 1616
Krzywicki, A. ... 1246
Ku, Y. H. ... 1600
Kumar Pandit, Lalit ... 1558
Kuratowski, K. ... 1293
Kursunoğlu, B. ... 1568
Kurzweil, J. ... 1022
Kustaanheimo, P. ... 1268, 1269
Kusunoki, Y. ... 967
Kuttner, B. ... 1131
Kuzmin, G. G. ... 1577
Kuz'mina, G. V. ... 977
Laasonen, P. ... 1028
Ladyženskaya, O. A. ... 1053, 1091, 1092
Laitoch, M. ... 1020, 1021
Lambek, J. ... 888
Lamperti, J. ... 1372
Lamprecht, E. ... 847
Landis, E. M. ... 1036
Lang, S. ... 1109
Langseth, A. ... 1552
Larionov, B. A. ... 1412
Lascu, A. T. ... 868
Laugwitz, D. ... 1283
Laurent, B. E. ... 1536
de La Vallée Poussin, C. ... 976
Lebed', G. K. ... 1165
Lebedev, V. I. ... 1101
de Leeuw, K. ... 1235
Lefschetz, S. ... 1005, 1320
Lehrer, T. A. ... 1365
Leĭfman, L. Ya. ... 952
Lenz, H. ... 1264, 1283
Leont'ev, A. F. ... 983
Levin, A. A. ... 998
Levin, B. Ya. ... 971
Levitan, B. M. ... 1078
Libois, P. ... 1270
Lichnerowicz, A. ... 996
Lidskiĭ, V. B. ... 1221
Lighthill, M. J. ... 1496
Likeš, J. ... 1393, 1394
Linfoot, E. H. ... 1519
Linnik, Yu. V. ... 831
Lohmander, B. ... 1427
Loinger, A. ... 1448
Lomsadze, G. A. ... 841
Lopatins'kiĭ, Ya. B. ... 1074
Łopuszański, J. ... 1534
Lotkin, M. ... 1401
Lüders, G. ... 1538
Lugaresi, E. ... 1527
Lun'kin, Yu. P. ... 1499
Lykova, O. B. ... 1046
Lyubič, Yu. I. ... 1179
MacDonald, D. K. C. ... 1450
McGregor, J. ... 1363
MacLane, S. ... 892
Maak, W. ... 1166
Mager, A. ... 1501
Magner, T. F. ... 1617
Malferrari, A. ... 1115
Malgrange, B. ... 1338, 1340
Mangeron, D. ... 1439
Manning, I. ... 1452
Marinescu, G. ... 1188, 1189, 1208
Markušević, L. A. ... 1206
Maroni, A. ... 874
Martin, A. ... 1544
Martinek, J. ... 1077
Martinelli, E. ... 993, 994
Maruašvili, T. I. ... 1421
Marziani, M. ... 1524
Maslennikova, V. N. ... 1058
Massey, F. J., Jr. ... 1376
Matsusaka, T. ... 878
Matsushima, Y. ... 995
Matveev, I. V. ... 1168
Mauldon, J. G. ... 1176
Maximon, L. C. ... 1443
Medgyessy, P. ... 1345
Meister, M. ... 1465
Melzi, G. ... 1324
Menchoff, D. E. ... 1156
Meng, Ching-Hwa ... 1223
Mercer, R. J. ... 1433
Meredith, C. A. ... 819
Mergelyan, S. N. ... 1146
Mesis, A. V. ... 961
Meyer, R. E. ... 1506
Michael, J. H. ... 955
Michel, P.-H. ... 830
Michiwaki, Y. ... 988
Mihallov, V. ... 1071
Mihailović, D. ... 1576
Mihlin, S. G. ... 1075
Mikusiński, J. ... 1214, 1230
Miller, K. S. ... 1371
Miščenko, E. F. ... 1029b
von Mises, R. ... 1504
Mishoe, L. I. ... 1154
Mishra, R. S. ... 1334
Misra, M. L. ... 1158
Mitrinović, D. S. ... 1049
Mitropol'skiĭ, Yu. A. ... 1030
Mitsui, T. ... 842
Modenov, P. S. ... 1267
Moór, A. ... 1573
Morgan, S. P. ... 1518
Morita, K. ... 894
Morpurgo, G. ... 1554
Moulton, F. R. ... 1003
Mróz, Z. ... 1479
Murray, J. D. ... 1490
Murthy, M. N. ... 1378
Mycielski, J. ... 934, 935

(*Continued on cover 4*)

AUTHOR INDEX

(Continued from cover 3)

Myasovskih, I. P. 1419
Nagaev, S. V. 1355
Nagata, J. 1298
Nagata, M. 853, 862, 876
Nagy, K. 1563
Naimark, M. A. 1240
Nakai, Y. 866
Nakamura, K. 903
Namioka, I. 1193
Napolitano, L. G. 1531
Narain, R. 1174, 1175
Nariboli, G. A. 1467
Nassif, M. 986
Natanson, G. I. 949
Natanson, I. P. 949, 1171
Naugol'nyh, K. A. 1508
Nefed'ev, G. N. 1275
Nevanlinna, R. 1081
Neyman, J. 1374
Nichols, W. 1556
Nicolescu, M. 1089
Nikodým, O. M. 1224
Nikolaev, P. V. 1425a-b
Nikol'skiĭ, S. M. 1111
Nilov, G. N. 941, 959
Nordes, A. P. 1271, 1614
Novožilov, V. V. 1480
Obrechkoff, N. 849
Ogiewetzki, I. I. 1155
Ohocimskiĭ, D. E. 1422
Ohtsuka, M. 1117, 1118
O'Keeffe, J. 1209, 1210
Okiljević, B. 1007
Oleĭnik, O. A. 1054, 1055, 1056, 1057
Olszak, W. 1478, 1479
Olver, F. W. J. 1012
Ono, T. 880
Orlicz, W. 1135, 1198
Osborn, H. 1073
Osgood, W. F. 936
Osserman, R. 968
Owens, O. G. 1076
Pachner, J. 1572
Panov, D. Yu. 1095
Papy, G. 1307
Parasiuk, O. S. = Paratyuk, O. S.
Paratyuk, O. S. 1535, 1567
Pardubský, B. 1395
Parhomenko, A. S. 1267
Pârvu, M. P. 1190, 1191
Pati, Tribikram. 1161
Patrauleu, N. N. 1505
Pavel, M. = Pârvu, M. P.
Pavlović, D. 1610
Perčinkova, D. 1059
Perez, A. 1595a-b
Peschl, E. 957
Pesonen, E. 1202
Petrovskiĭ, I. G. 1036
Phakadze, S. S. 946
Piccard, S. 902
Pickert, G. 1263
Pirl, U. 966
Piszczek, K. 1468
Plainevaux, J. E. 1413
Pliś, V. A. 1041
Plotkin, B. I. 921
Pócsik, G. 1542
Podderyugin, V. D. 911
Pokorná, O. 1407
Pokornyi, V. V. 1183
Pontryagin, L. S. 1029a
Popescu, I. L. 1489
Postnikov, M. M. 1311, 1312
Press, F. 1475
Prvanović, M. 1333
Pták, V. 1198
Putnam, C. R. 1026
Pyateckiĭ-Šapiro, I. I. 991
Ráb, M. 1023
Radó, T. 953
Radu, N. C. 1144
Rapcsák, A. 1341
Raševskiĭ, P. K. 929
Ree, R. 906, 912
Ree, T. 1509
Reiter, H. 1170a-b
Rényi, A. 1370
Rešetnyak, Yu. G. 1279
Reuter, G. E. H. 1004
Ridder, J. 950
Rider, P. R. 1379
Rittstein, S. 1427
Roberson, R. E. 1579, 1580, 1585
Roberts, P. H. 1513
Robinson, R. M. 832
Rodniansky, A. M. 1295
Rogers, F. E. 1532
Rogožin, V. S. 964
Rohleder, H. 1602, 1603
Rohlin, V. A. 1318
Röhrl, H. 992
Roşculet, M. N. 1102
Rose, A. 818
Rosenblum, M. 1139
Rosenstock, H. B. 1586
Rosina, B. A. 869, 870
Rosman, H. 1525
Rosser, J. B. 818
Rossow, V. J. 1510
Rothe, E. H. 1244
Rozenfel'd, B. A. 1614
Rosenknop, I. Z. 896
Rubin, H. 1386
Rubinštein, G. Š. 1219
Ruchte, M. F. 913
Rudin, M. E. 1286
Rudin, W. 1163
Rühs, F. 900
Russian-English glossary of acoustics and ultrasonics. .. 1618
Russian-English glossary of solid state physics. 1619
Rutishauser, H. 1134
Ryshkov, S. S. 1303, 1304
Ržanicyn, A. R. 1483
Rzewuski, J. 1246
Sabidussi, G. 1322
Sadovskiĭ, L. E. 916
Sahnovič, L. A. 1228
Sakuma, M. 855
Salmon, G. 1265
Saltykov, N. 1072
Saminsky, L. 816
Šapiro, Z. Ya. 1061
Sargan, J. D. 1587
Sario, L. 979
Sasayama, H. 1273
Satô, Y. 1471
Sauer, R. 1503
Săvulescu, S. 1495
Scarf, H. E. 1276
Schaefer, H. 1184
Schaeffer, A. C. 972
Schäffer, J. J. 1033
Schiffer, M. M. 965
Schlesinger, J. W. 1316
Schmauch, H. 1526
Schmetterer, L. 1387
Schubert, G. U. 1526
Schücking, E. 1584
Schwartz, C. 1562
Schwinger, J. 1539
Scott, W. R. 918
Seligman, G. B. 891
Seres, I. 861
Sestini, G. 1086
Severi, F. 875
Shafarevich, I. R. 867, 881
Sharma, J. L. 1000
Shell, D. L. 1402
Shepperd, J. A. H. 910
Shibata, T. 1211
Shibuya, T. 1201
Shimada, N. 1317
Shmulian, Y. L. = Šmul'yan, Yu. L.
Siddiqi, J. A. 1180
de Siebenthal, J. 926
Sigalov, A. G. 1245
Sikorski, R. 1214
Silin, V. P. 1547
Silverman, R. A. 1373
Silvey, S. D. 1382
Simonov, M. I. 1094
Simonsen, W. 1122
Singh, S. K. 974
Singh, U. N. 1159
Skačkov, B. N. 1014
Sklar, A. 838
Skorohod, A. V. 1351
Slugin, S. N. 1180
Smale, S. 1319
Smirnov, G. P. 836
Smirnov, M. M. 1181
Smirnov, S. V. 1425
Smith, H. F. 836
Šmul'yan, Yu. L. 1197, 1233
Smygun, A. N. 1132
Sobol', I. M. 1431
Sobolev, S. L. 1196
Sobolevskiĭ, P. E. 1050
Solov'ev, V. G. 1557
Spain, B. 1262
Sparenberg, J. A. 1181
Spitzbart, A. 1402
Šreĭder, Yu. A. 1238
Standish, C. 1150
Steger, A. 844
Stein, E. M. 1216
Stein, S. K. 922
Steinberg, R. 1260
Šteingauz, G. = Steinhaus, H.
Steinhaus, H. 1148, 1375
Sterne, T. E. 1578
Stoka, M. I. 928
Straneo, P. 1569
Štraus, A. V. 1067
Stroh, A. N. 1469
Strohl, G. R., Jr. 1287
Stupina, I. D. 826
Su, Buchin. 1328, 1335, 1332
Sunakawa, S. 1541
Sundström, R. 1473
Suprunenko, D. A. 908
Svoboda, A. 1411
Swingle, P. M. 1285
Szász, F. A. 857, 885
Szegö, G. 1349
Szmydt, Z. 1082, 1083
Tachikawa, H. 894, 895
Takács, L. 1368, 1369
Takeuchi, K. 897
Tallini, G. 865
Talypov, G. B. 1482
Tanikawa, Y. 1540
Taussky, O. 845
Taylor, A. E. 1225
Taylor, S. J. 1364
Tchakaloff, V. 1145
Teleman, S. 1308
Teodorescu, N. 1080
Terracini, A. 1261
Thionet, P. 1383, 1384, 1385
Thompson, G. T. 1423
Thüring, B. 1432
Tihonov, V. I. 1599
Timan, A. F. 937
Tisza, L. 1452
Titchmarsh, E. C. 1065
Todd, John. 1408
Tolsted, E. 1110
Tricomi, F. G. 1177
Truesdell, C. 1437
Tsuji, K. 1236
Tukey, J. W. 1381
Tupper, B. O. J. 1583
Turán, P. 945
Turner, L. H. 954
Turri, T. 882, 883
Uchiyama, S. 866
Uflyand, Ya. S. 1450
Ulin, B. 1248
Uljanov, P. L. 1157
Urbanik, K. 951
Vallander, S. V. 1106
Van Hove, L. 1449
Varadarajan, V. S. 1347, 1348
Varšamov, R. R. 1596
Vašarin, A. A. 1113
Vazsonyi, A. 1586
Vejvoda, O. 1044
Vekua, N. P. 980
Venkov, B. A. 1392
Verma, P. D. 1493
Vermes, R. 1149
Vilenkin, N. J. 998
Vincensini, P. 1328
Vinrogadov, A. I. 831
Vinokurov, V. G. 1354
Višik, M. I. 1091, 1092
Viswanathan, K. S. 1549
Vitásek, E. 1420
Vlasova, Z. P. 1422
Voelker, D. 1529
Vollokov, M. I. 1035
Volkov, A. N. 1472
Volkov, I. I. 1137
Volkov, V. I. 1205
Volosov, V. M. 1048a-b
Voronovskaya, E. V. 1181
Vostrecov, B. A. 1096
Vurcfeld, V. 1435
Vvedenskaya, A. V. 1485
Vvedenskaya, N. D. 1054
Vyčichlo, F. 1612
Wagner, R. 1266
Wallace, A. D. 823, 824, 825
Wallace, A. H. 873
Walz, A. 1497
Wan, Cheh-hsiao. 909
Warner, F. J. 1418
Watanabe, S. 1540
Wegner, U. 1457
Weier, J. 1314
Weiss, G. 1216
Weizmann, A. 1330
Welch, L. R. 1597
Wendel, J. G. 1344
Wergeland, H. 1443
Weston, J. D. 1288
Whitehead, G. W. 1306
Whyburn, G. T. 1299, 1300
Wienholtz, E. 1090
Wilhelmsson, H. 1522
Williams, G. T. 1390
Wintner, A. 990, 1009, 1043
Witting, H. 1498
Wright, E. M. 828
Wylie, C. R., Jr. 884
Yaglom, A. M. 1353
Yaglom, I. M. 1614
Yakubović, V. A. 1045
Yamaguchi, R. 1471
Yamanaka, T. 1212, 1213
Yamanoshita, T. 1310
Yano, K. 1142
Yastrebov, Yu. M. 1233
Yeh, G. C. K. 1077
Yien, Sze-chien. 907
Young, R. C. 1543
Zablockaya, N. I. 1132
Zăgănescu, M. 1574
Zaharov, V. K. 1114, 1207
Zakharov, V. K. = Zaharov, V. K.
Zaĭcev, M. N. 973
Zalgaller, V. A. 1277
Zamfirescu, I. 1009
Zamorski, J. 1246
Zappa, G. 915
Zeeman, E. C. 1309
Zelinsky, D. 889
Zerner, M. 1084
Zeuli, T. 1530
Zięba, A. 1246
Ziegler, H. 1451
Zierler, N. 851
Zil'berman, G. E. 1559
Zislin, G. M. 1548
Zítek, F. 1367
Žitomirskiĭ, Ya. I. 1084
Zubov, V. I. 1006, 1034
Zuhovickiĭ, S. I. 1204
Zumino, B. 1539
Zwinggi, E. 1581